Regular Expressions

Recap from Last Time

Automata Progression

- **DFAs** (Deterministic Finite Automata)
  = **NFAs** (DFA + superpowers of guessing)
- **PDAs** (NFA + a stack to use as memory)
  We'll actually explore an equivalent mechanism called CFGs, you'll see soon...
- **Turing Machines** (DFA or NFA + an infinite array to use as memory)

New

At each step, we'll try to find a problem (language) that's too hard for the existing machine type, and motivate moving up a level in power.

Language Concatenation

- If $w \in \Sigma^*$ and $x \in \Sigma^*$, then wx is the ***concatenation*** of w and x.
- If L_1 and L_2 are languages over Σ, the ***concatenation*** of L_1 and L_2 is the language L_1L_2 defined as

$$\boldsymbol{L_1L_2 = \{\, wx \mid w \in L_1 \text{ and } x \in L_2 \,\}}$$

- Example: if $L_1 = \{$ **a**, **ba**, **bb** $\}$ and $L_2 = \{$ **aa**, **bb** $\}$, then

$$L_1L_2 = \{ \textbf{aaa}, \textbf{abb}, \textbf{baaa}, \textbf{babb}, \textbf{bbaa}, \textbf{bbbb} \}$$

Lots and Lots of Concatenation

- Consider the language L = { **aa**, **b** }
- $L^0 = \{\varepsilon\}$
- $LL = L^2$ is the set of strings formed by concatenating pairs of strings in L.

  { **aaaa**, **aab**, **baa**, **bb** }
- $LLL = L^3$ is the set of strings formed by concatenating triples of strings in L.

  { **aaaaaa**, **aaaab**, **aabaa**, **aabb**, **baaaa**, **baab**, **bbaa**, **bbb**}
- $LLLL = L^4$ is the set of strings formed by concatenating quadruples of strings in L.

  { **aaaaaaaa**, **aaaaaab**, **aaaabaa**, **aaaabb**, **aabaaaa**,
  aabaab, **aabbaa**, **aabbb**, **baaaaaa**, **baaaab**, **baabaa**,
  baabb, **bbaaaa**, **bbaab**, **bbbaa**, **bbbb**}

The Kleene Closure

- An important operation on languages is the ***Kleene Closure***, which is defined as

$$L^* = \{ w \in \Sigma^* \mid \exists n \in \mathbb{N}.\ w \in L^n \}$$

Closure Properties

- ***Theorem:*** If L_1 and L_2 are regular languages over an alphabet Σ, then so are the following languages:
  - $\overline{L}_1$
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - $L_1 L_2$
  - L_1^*
- These properties are called ***closure properties of the regular languages***.

Quick check 1:
Let Σ = {1, 2, 3, a, b, c}.
Let L_1 = {aa, b}, L_2 = {33, 2}
be languages over Σ.

Name two strings string **in** $\overline{L}_1$.

Closure Properties

- ***Theorem:*** If L_1 and L_2 are regular languages over an alphabet Σ, then so are the following languages:
  - $\overline{L_1}$
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Quick check 2:
Let Σ = {1, 2, 3, a, b, c}.
Let L_1 = {aa, b}, L_2 = {33, 2}
be languages over Σ.

Name two strings **in** $L_1 \cup L_2$.

Closure Properties

- ***Theorem:*** If L_1 and L_2 are regular languages over an alphabet Σ, then so are the following languages:
  - $\overline{L_1}$
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - $L_1 L_2$
  - L_1^*
- These properties are called ***closure properties of the regular languages***.

Quick check 3:
Let Σ = {1, 2, 3, a, b, c}.
Let L_1 = {aa, b}, L_2 = {33, 2}
be languages over Σ.

Name two strings **in** $L_1 \cap L_2$.

Closure Properties

- ***Theorem:*** If L_1 and L_2 are regular languages over an alphabet Σ, then so are the following languages:
  - $\overline{L_1}$
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - $L_1 L_2$
  - L_1^*
- These properties are called ***closure properties of the regular languages***.

Quick check 4:
Let Σ = {1, 2, 3, a, b, c}.
Let L_1 = {aa, b}, L_2 = {33, 2}
be languages over Σ.

Name two strings **in** $L_1 L_2$.

Closure Properties

- ***Theorem:*** If L_1 and L_2 are regular languages over an alphabet Σ, then so are the following languages:
  - $\overline{L_1}$
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
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- These properties are called ***closure properties of the regular languages***.

Quick check 5:
Let Σ = {1, 2, 3, a, b, c}.
Let L_1 = {aa, b}, L_2 = {33, 2}
be languages over Σ.

Name two strings **in** L_1^*.

Defining Regular Languages

- We currently have several tools for showing a language L is regular:
  - Construct a **DFA** for L.
    - *We used this for a variety of languages, as well as proving closure for complement.*
  - Construct an **NFA** for L.
    - *We used this for a variety of languages, as well as proving closure for union, concatenation, and Kleene *.*
  - Combine several simpler regular languages together via **closure properties** to form L.
    - *We used this for proving closure for intersection.*
    - *(Today we'll greatly expand on this last idea.)*

New Stuff!

Another View of Regular Languages:

Regular Expressions (regex)

xkcd

80 minutes from now, you will be able to say this

Regular Expressions

- ***Regular expressions*** are a way of describing a language via a string pattern-match representation.
  - They’re used all over in practical day-to-day programming:
    - Data validation tools in just about every programming language (python, Javascript, C++, etc)
    - UNIX `grep` and `flex` tools to search files
    - Compilers

Constructing Regular Languages with RegEx

- ***Idea:*** Build up all regular languages as follows:
  - Start with a small set of super-simple languages we already know to be regular.
  - Using closure properties, combine these simple languages together to form more languages.
  - Because we only use closure properties, the results of even the most elaborate combinations are guaranteed to be regular.
- *This is a bottom-up approach to the regular languages.*

Constructing Regular Languages with RegEx

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  - Using clos
    languages
  - Because w
    of even th
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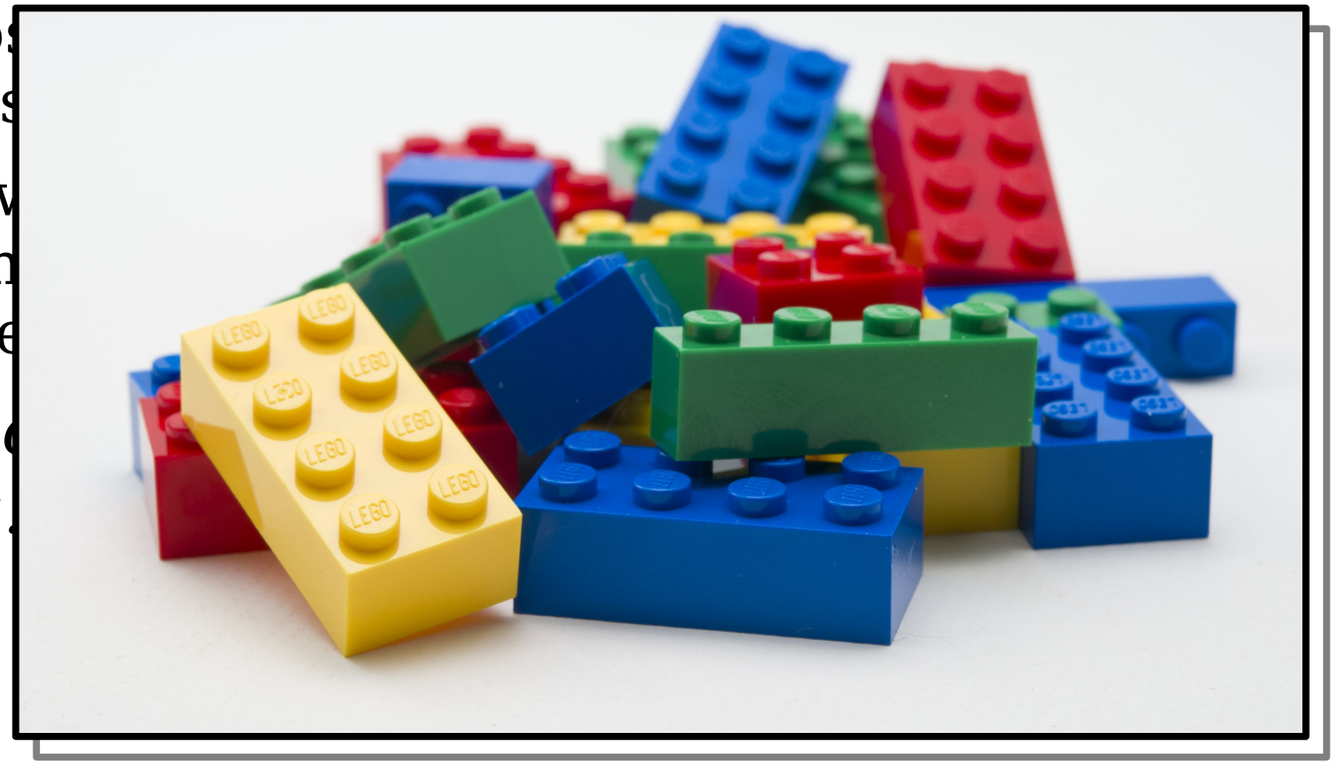
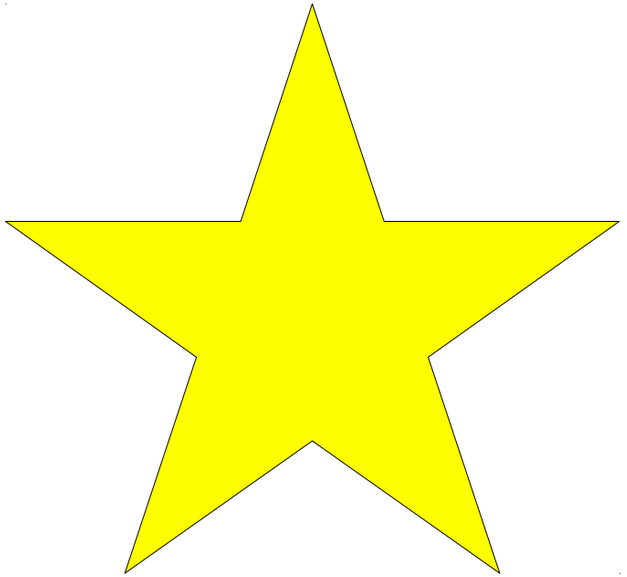
- *This is a be*
  languages.



RegEx “Starter Languages”

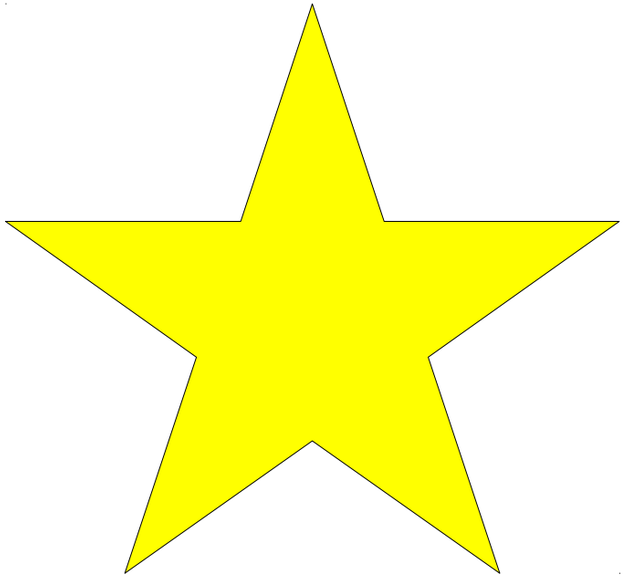
- The regular expressions begin with **three starter languages** *(a bit like base cases in induction!)*

  1. The symbol **Ø** is a regular expression that represents the empty language Ø.
  2. For any **a** ∈ Σ, the symbol **a** is a regular expression for the language {**a**}. (*So this “starter language” is actually many languages, one for each character in Σ.)*
  3. The symbol **ε** is a regular expression that represents the language {ε}.



Combining RegEx

- Let R_1 and R_2 be regular expressions (either one of the starter languages, or a more complex regex). We can combine them in these ways:
- ***Concatenation:*** $\boldsymbol{R_1R_2}$
- ***Union:*** $\boldsymbol{R_1 \cup R_2}$
- ***Kleene closure:*** $\boldsymbol{R_1*}$
- ***Parentheses:*** $\boldsymbol{(R_1)}$ may be used to control order of operations, and for reading clarity.



Operator Precedence

- Here's the operator precedence for regular expressions:

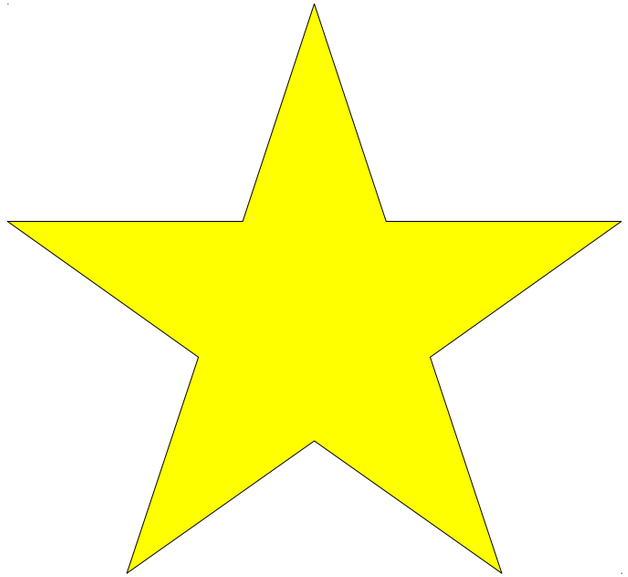
$$(R_1)$$

$$R_1*$$

$$R_1R_2$$

$$R_1 \cup R_2$$

- So **ab*c∪d** is parsed as **((a(b*))c)∪d**



Regular Expression Examples

- The regular expression **`trick∪treat`** represents the language

  { **`trick`**, **`treat`** }.

- The regular expression **`booo*`** represents the regular language

  { **`boo`**, **`booo`**, **`boooo`**, … }.

- The regular expression **`candy!(candy!)*`** represents the regular language

  { **`candy!`**, **`candy!candy!`**, **`candy!candy!candy!`**, … }.

Regular Expressions, Formally

- The ***language of a regular expression*** is the language described by that regular expression.
- Formally:
  - $\mathscr{L}(\mathbf{\varepsilon}) = \{\varepsilon\}$
  - $\mathscr{L}(\mathbf{\emptyset}) = \emptyset$
  - $\mathscr{L}(\mathbf{a}) = \{\mathbf{a}\}$
  - $\mathscr{L}(R_1 R_2) = \mathscr{L}(R_1)\,\mathscr{L}(R_2)$
  - $\mathscr{L}(R_1 \cup R_2) = \mathscr{L}(R_1) \cup \mathscr{L}(R_2)$
  - $\mathscr{L}(R^*) = \mathscr{L}(R)^*$
  - $\mathscr{L}(\mathbf{(}R\mathbf{)}) = \mathscr{L}(R)$

Regex quick check:

Let Σ = {a, b, c, d}.

Let $L_1 = \mathcal{L}(\mathbf{a(b \cup c)((d))})$ be a language over Σ.

Name two strings **in** L_1.

Name one string **not in** L_1.

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains **aa** as a substring $\}$.

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$$(\mathbf{a} \cup \mathbf{b})^*\mathbf{aa}(\mathbf{a} \cup \mathbf{b})^*$$

Designing Regular Expressions

- Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains $\mathtt{aa}$ as a substring $\}$.

$$(\mathtt{a} \cup \mathtt{b})^*\mathtt{aa}(\mathtt{a} \cup \mathtt{b})^*$$

Designing Regular Expressions

- Let $\Sigma = \{\mathbf{a}, \mathbf{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains **aa** as a substring $\}$.

$$(\mathbf{a} \cup \mathbf{b})^*\mathbf{aa}(\mathbf{a} \cup \mathbf{b})^*$$

bbabbbaabab

aaaa

bbbbbabbbbaabbbbb

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains **aa** as a substring $\}$.

(a ∪ b)*aa(a ∪ b)*

bbabbbaabab

aaaa

bbbbbabbbbaabbbbb

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains **aa** as a substring $\}$.

Σ*aaΣ*

bbabbbaabab

aaaa

bbbbbabbbbaabbbbb

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{\ w \in \Sigma^* \mid |w| = 4\ \}$.

Designing Regular Expressions

Let $\Sigma = \{\mathbf{a}, \mathbf{b}\}$.

Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

The length of a string w is denoted $|w|$

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
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Designing Regular Expressions

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ΣΣΣΣ

Designing Regular Expressions

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ΣΣΣΣ

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ΣΣΣΣ

aaaa
baba
bbbb
baaa

Designing Regular Expressions

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- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

ΣΣΣΣ

aaaa
baba
bbbb
baaa

Designing Regular Expressions

- Let $\Sigma = \{\mathbf{a}, \mathbf{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

$\mathbf{\Sigma^4}$

aaaa
baba
bbbb
baaa

Designing Regular Expressions

- Let $\Sigma = \{\mathbf{a}, \mathbf{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

$$\mathbf{\Sigma^4}$$

aaaa
baba
bbbb
baaa

Designing Regular Expressions

- Let $\Sigma = \{\mathbf{a}, \mathbf{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains at most one $\mathbf{a} \}$.

Here are some candidate regular expressions for the language L. **How many** of these are correct? (Discuss specifically which with your neighbors.)

$$\mathbf{\Sigma^*a\Sigma^*}$$

$$\mathbf{b^*ab^* \cup b^*}$$

$$\mathbf{b^*(a \cup \varepsilon)b^*}$$

$$\mathbf{b^*a^*b^* \cup b^*}$$

$$\mathbf{b^*(a^* \cup \varepsilon)b^*}$$

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{\ w \in \Sigma^* \mid w$ contains at most one **a** $\}$.

b*(a $\cup$ **ε)b***

Designing Regular Expressions

- Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains at most one $\mathtt{a} \}$.

$$\mathtt{b}^*(\mathtt{a} \cup \varepsilon)\mathtt{b}^*$$

Designing Regular Expressions

- Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains at most one $\mathtt{a} \}$.

$$\mathtt{b}^*(\mathtt{a} \cup \varepsilon)\mathtt{b}^*$$

bbbbabbb

bbbbbb

abbb

a

Designing Regular Expressions

- Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains at most one $\mathtt{a} \}$.

$$\mathtt{b}^\star(\mathtt{a} \cup \varepsilon)\mathtt{b}^\star$$

bbbbabbb

bbbbbb

abbb

a

Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{\ w \in \Sigma^* \mid w$ contains at most one **a** $\}$.

b*a?b*

bbbbabbb

bbbbbb

abbb

a

A More Elaborate Design

- Let Σ = { `a`, `.`, `@` }, where `a` represents "some letter."
- Let's make a regex for email addresses.

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cs103@cs.stanford.edu

first.middle.last@mail.site.org

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```
aa*(.aa*)*@
```

```
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```

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- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
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```
aa*(.aa*)*@aa*.aa*
```

```
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A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
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$$a^+(.aa^*)^*@aa^*.aa^*(.aa^*)^*$$

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A More Elaborate Design

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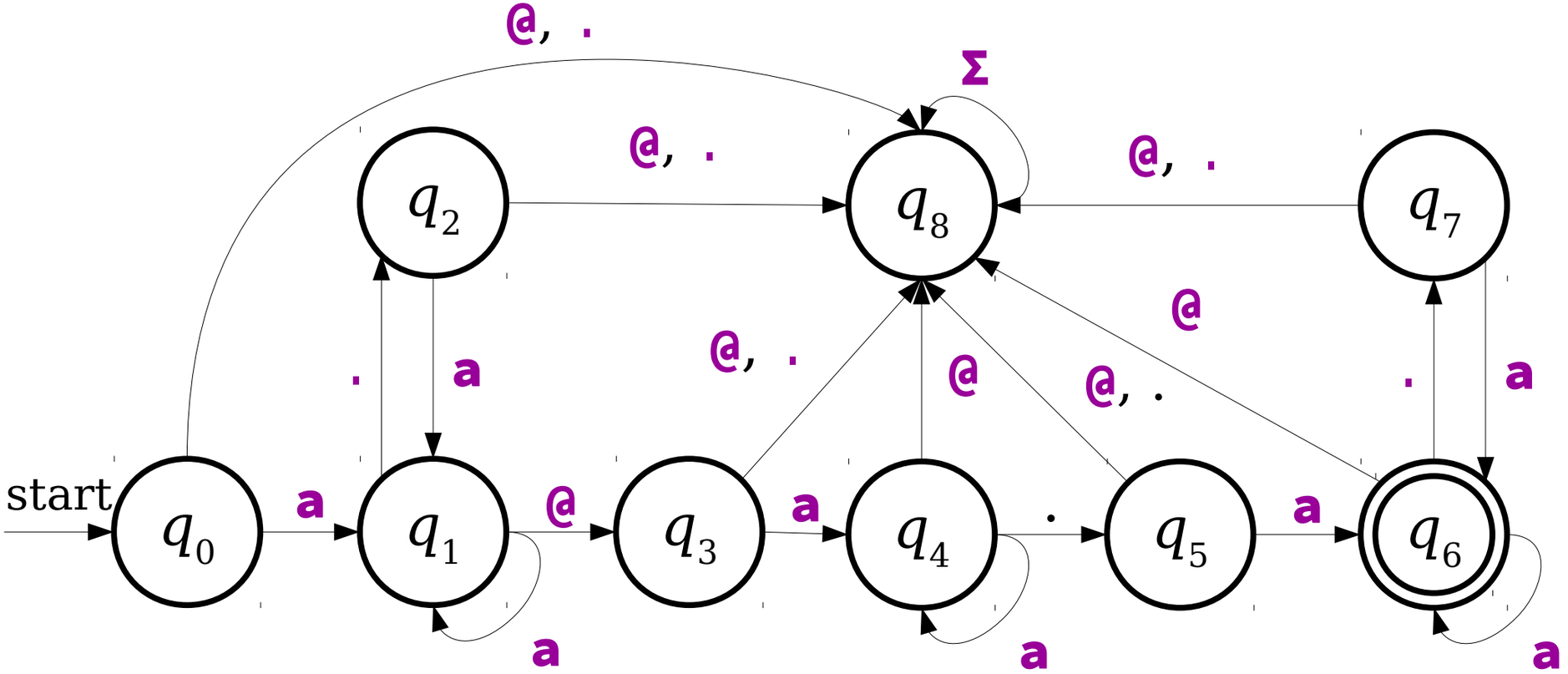
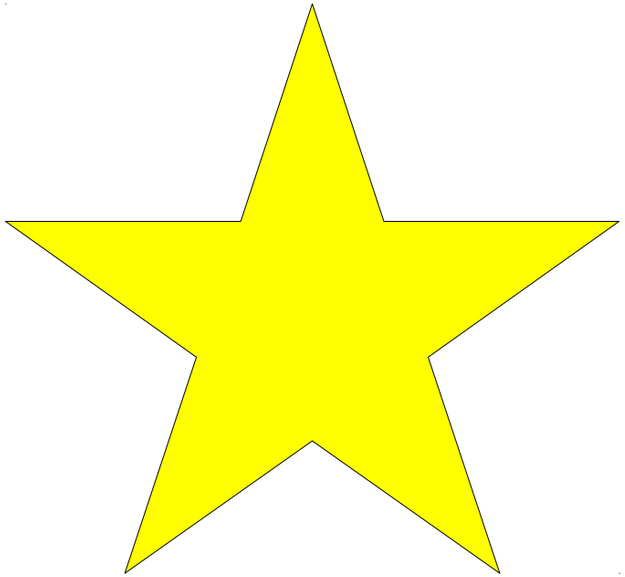
For Comparison

$$a^+(.a^+)^*@a^+(.a^+)^+$$

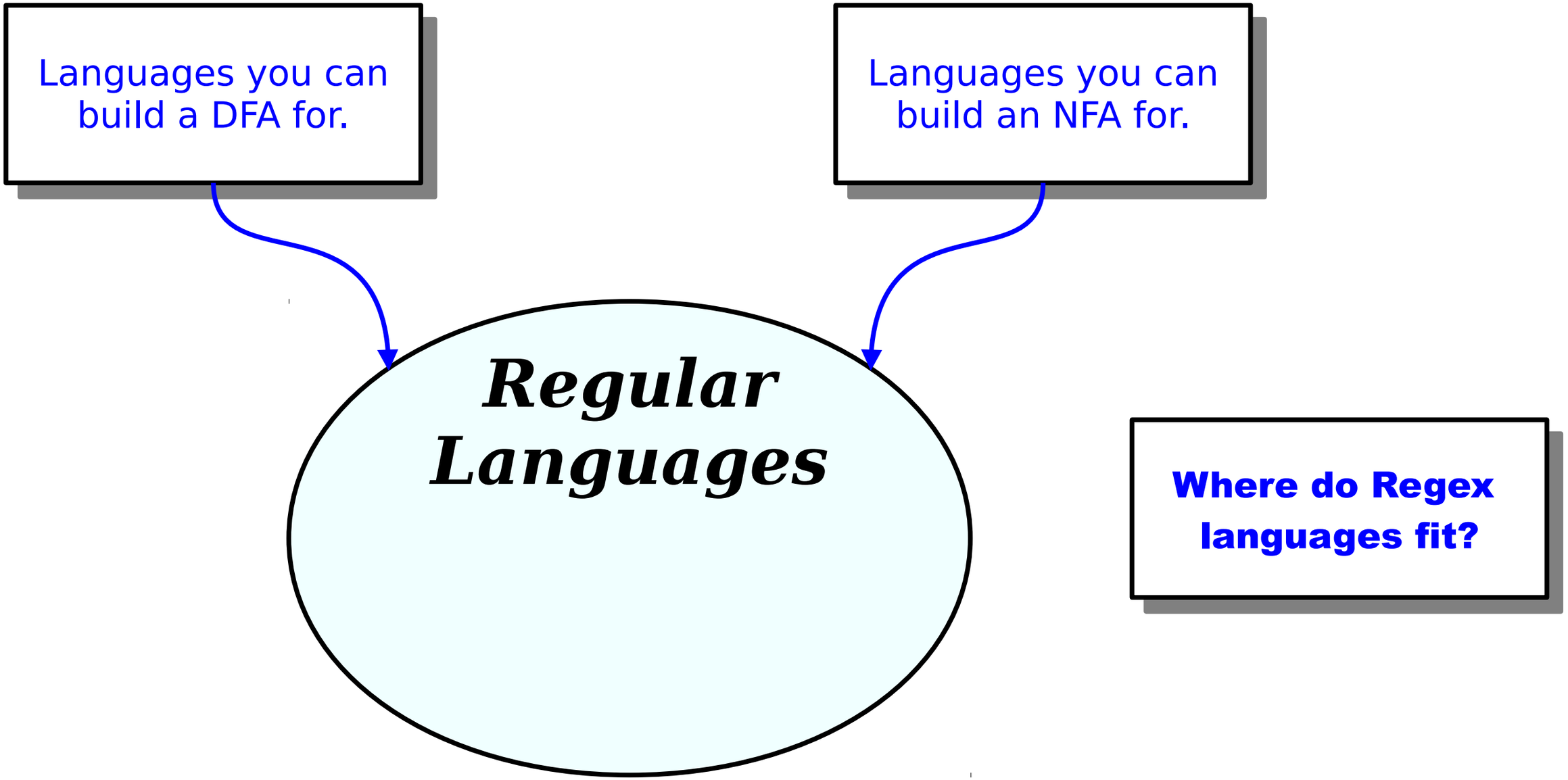


RegEx Shorthand Summary

- **R**^{n} is shorthand for **RR** … **R** (n times).
  - Edge case: define **R**^{0} = ε.
- **Σ** is shorthand for "any character in Σ."
- **R?** is shorthand for **(R ∪ ε)**, meaning "zero or one copies of R."
- **R**$^{+}$ is shorthand for **RR***, meaning "one or more copies of R."



The Lay of the Land


Languages you can build a DFA for.
Languages you can build an NFA for.
Regular Languages
Where do Regex languages fit?

The Power of Regular Expressions

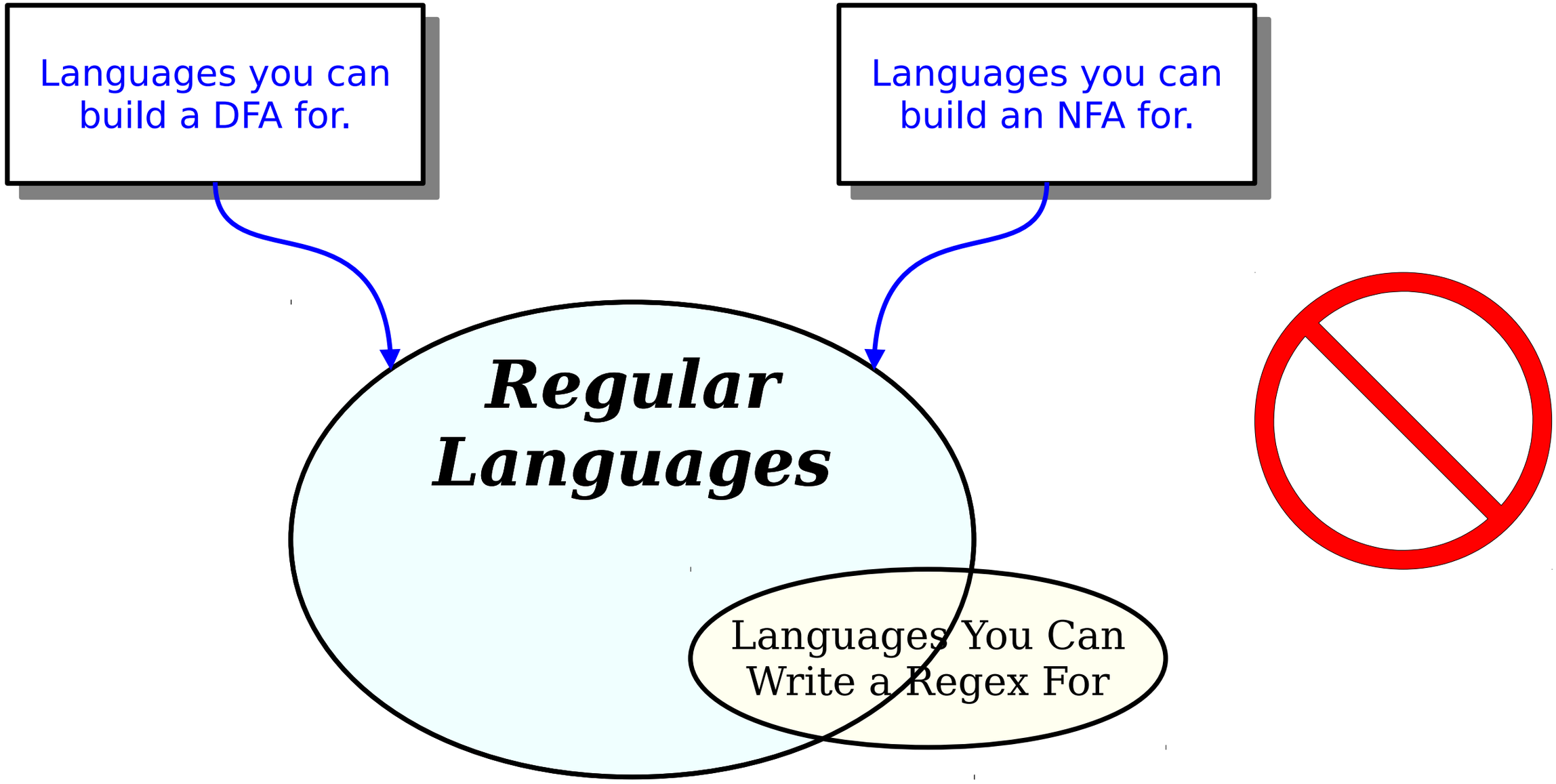
Theorem: If R is a regular expression, then $\mathscr{L}(R)$ is regular.

Proof idea: Use induction!

- **Base cases:** The atomic regular expressions all represent regular languages. *(Remember I said they were like induction base cases? Well they literally are!)*
- The combination steps represent closure properties.
- So anything you can make from them must be regular!

Thompson’s Algorithm

- In practice, many regex matchers use an algorithm called ***Thompson's algorithm*** to convert regular expressions into NFAs (and, from there, to DFAs).
  - Read Sipser if you’re curious!
- ***Fun fact:*** the “Thompson” here is Ken Thompson, one of the co-inventors of Unix!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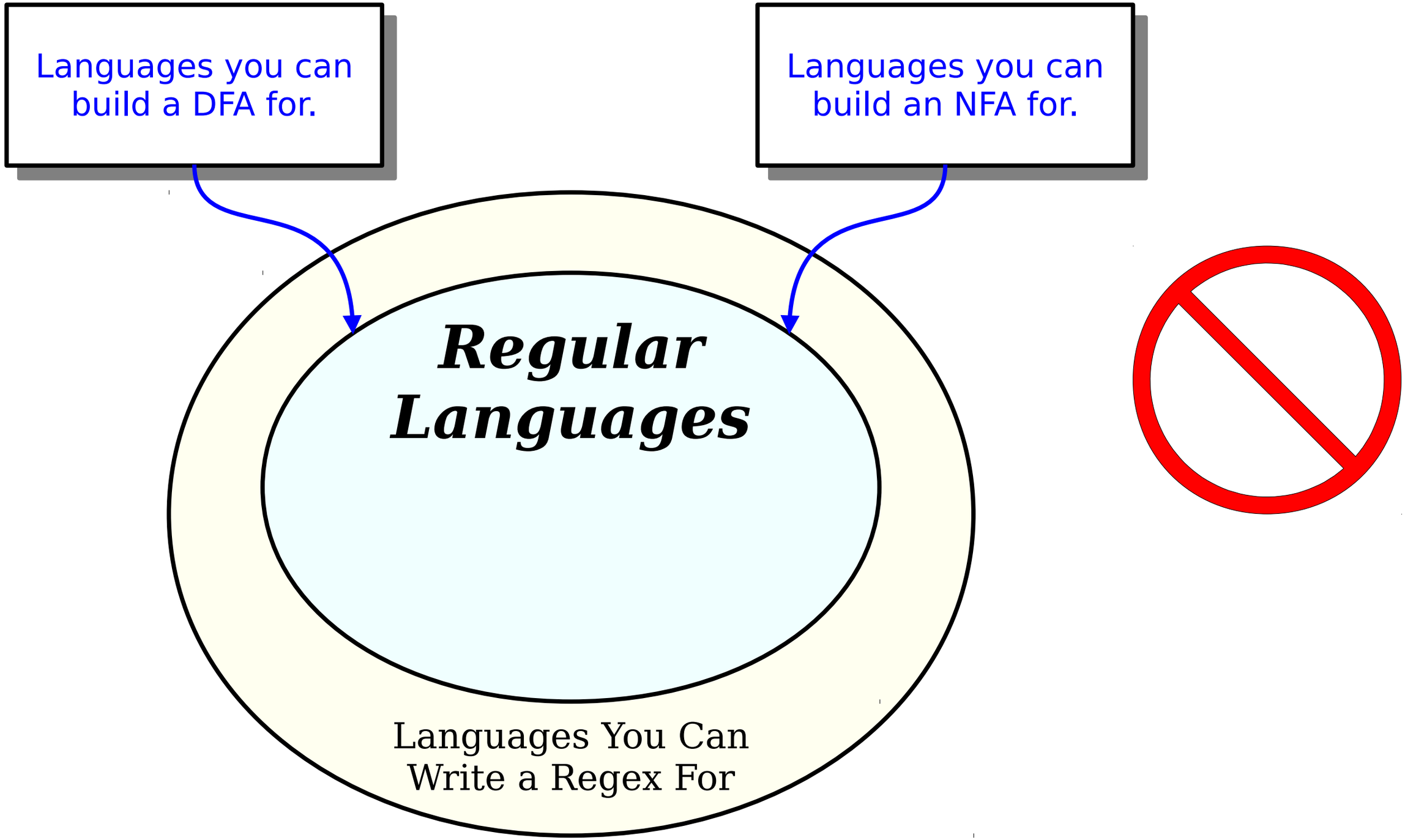
Regular Languages
Languages You Can Write a Regex For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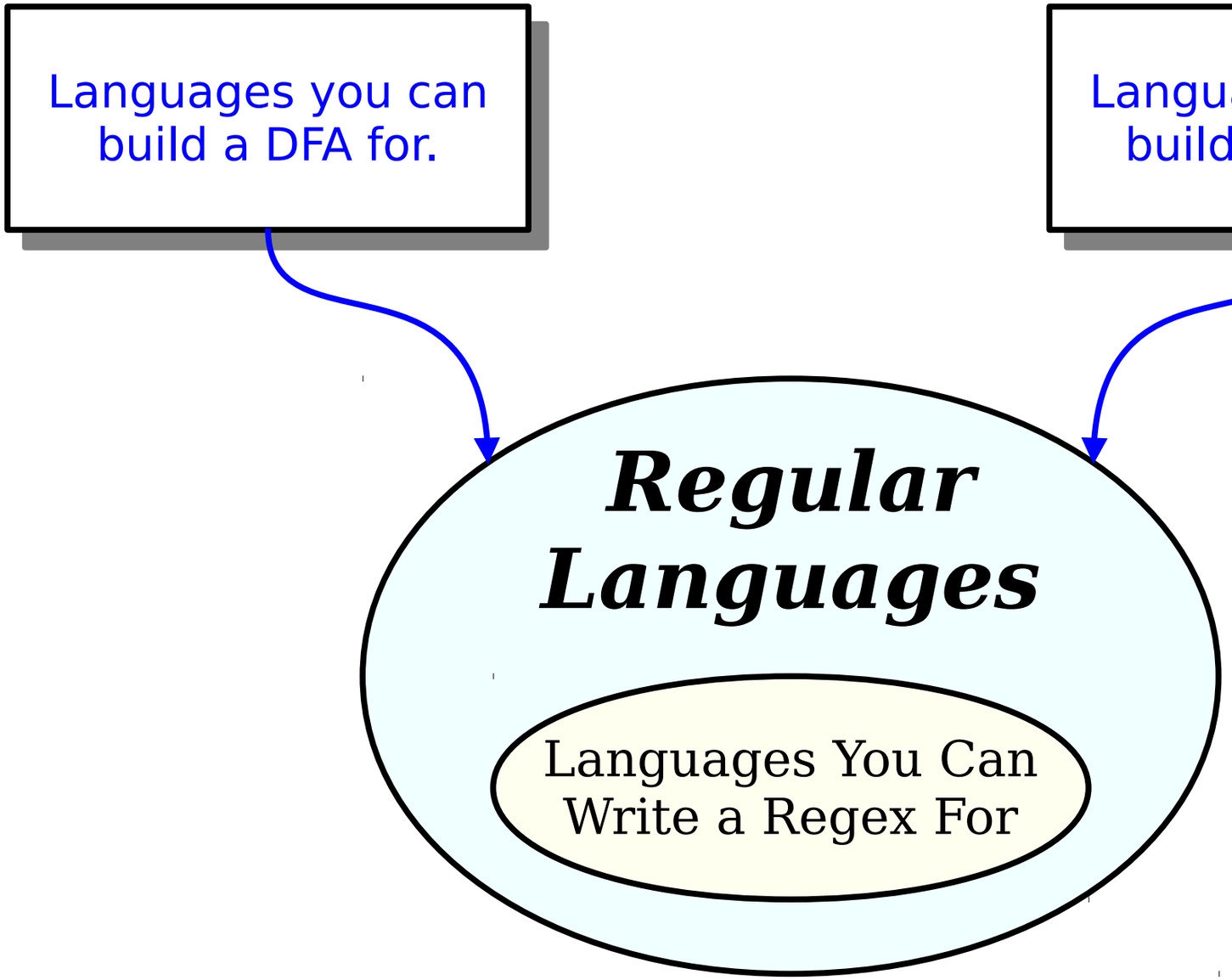
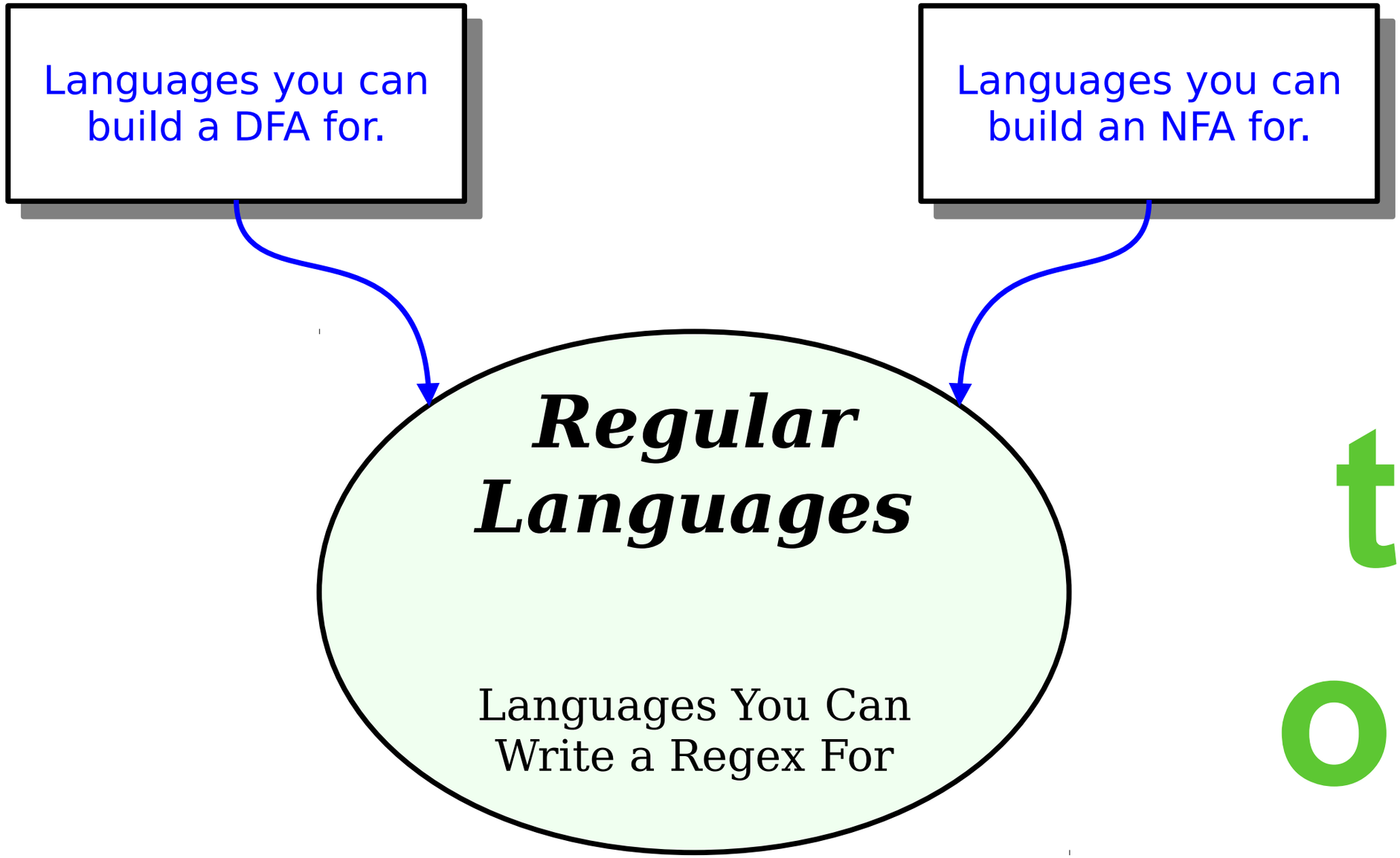
Regular Languages

Languages You Can Write a Regex For


Languages you can build a DFA for.
Languages you can build an NFA for.
Regular Languages
Languages You Can Write a Regex For

This one?

or


Languages you can build a DFA for.
Languages you can build an NFA for.
Regular Languages
Languages You Can Write a Regex For

this one?

The Power of Regular Expressions

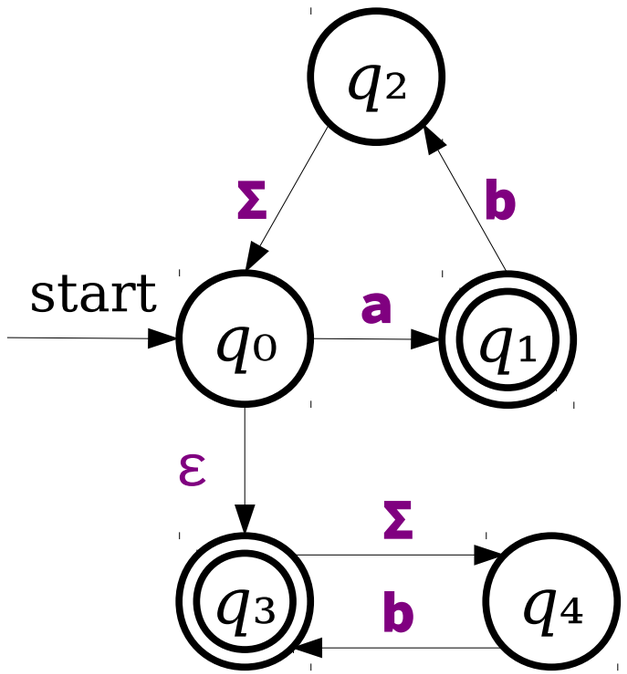
Theorem: For all languages L, if L is a regular language, then there is a regular expression for L.

This is not obvious!

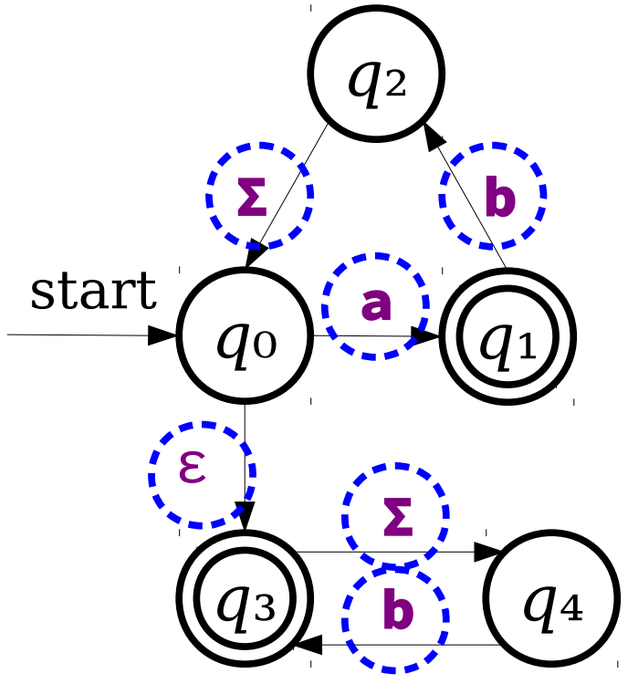
Proof plan:

- ***Assume:*** Pick an arbitrary regular language L.
- ***Want to show:*** We want to show there exists a regex for L.
- Since L is regular, there exists an NFA N for L.
- Describe how to transform N into a Regex, in a generalizable way. *For the next few slides, we'll explore how to do that.*

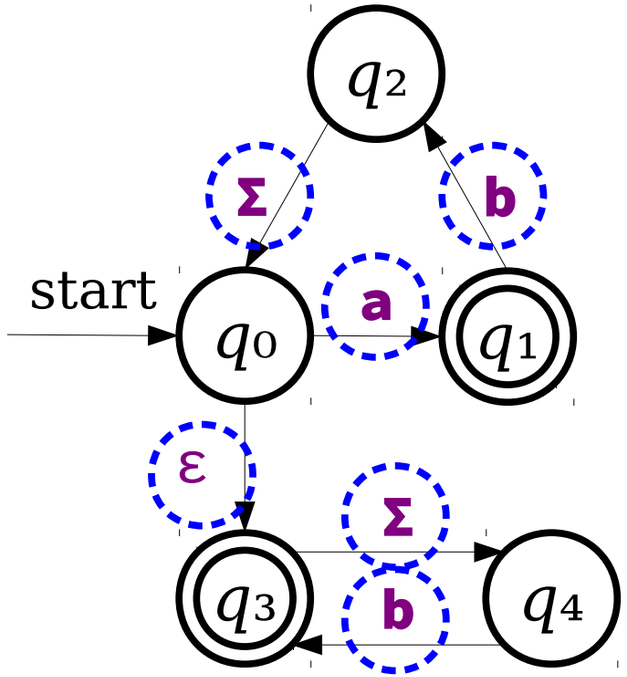
Generalizing NFAs



Generalizing NFAs

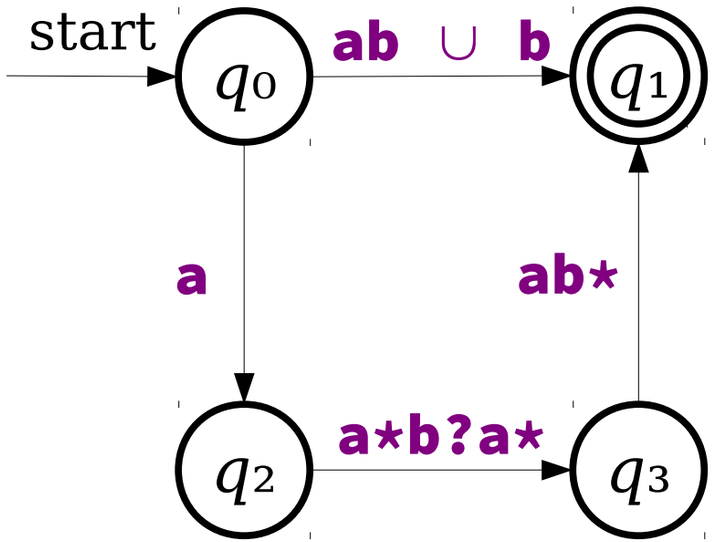


Generalizing NFAs

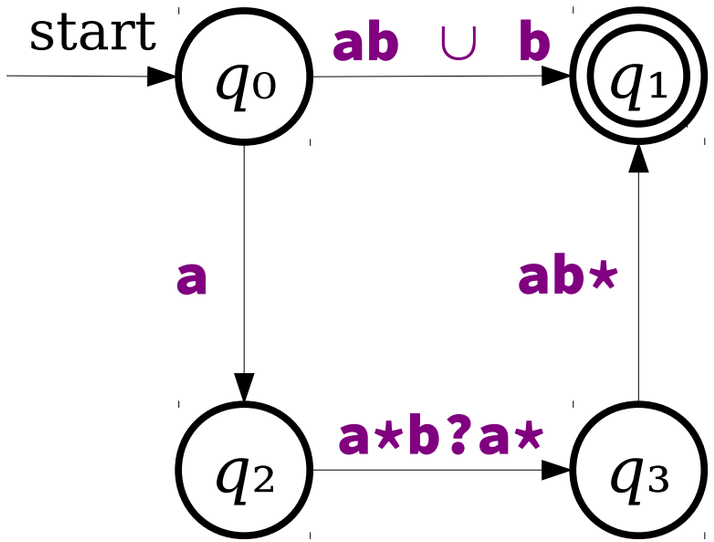


These are all regular expressions!

Generalizing NFAs

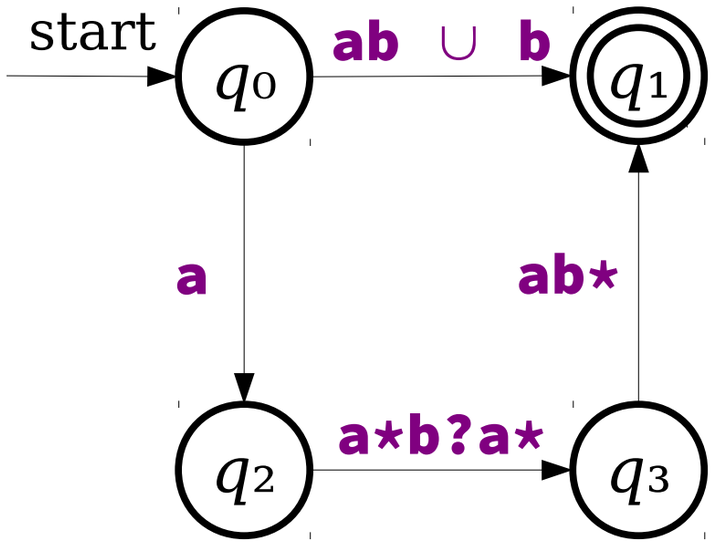


Generalizing NFAs



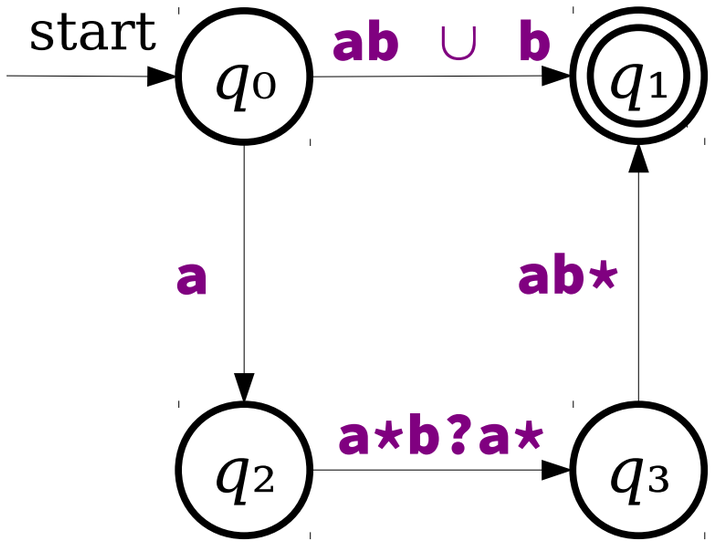
Note: Actual NFAs aren't allowed to have full regex transitions like these. This is just a thought experiment.

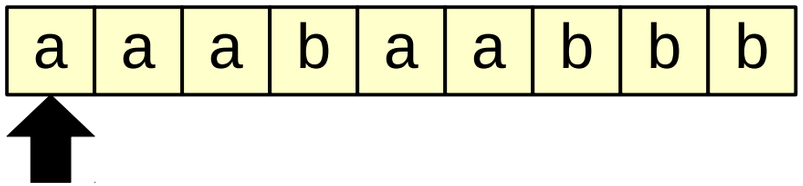
Generalizing NFAs



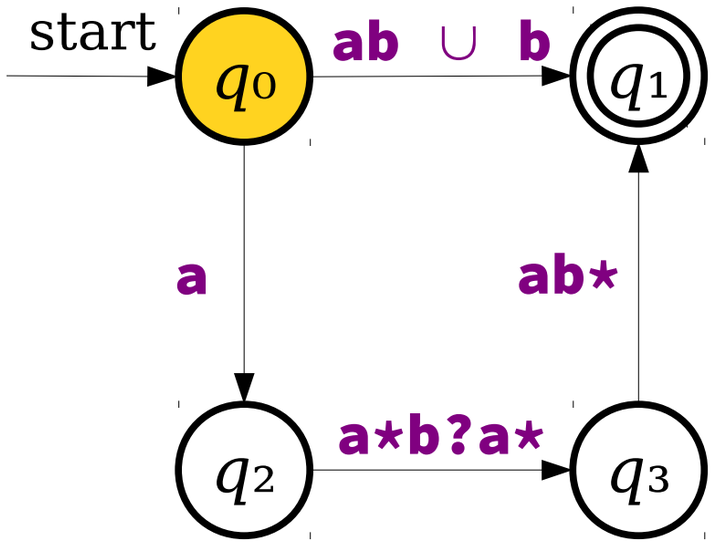
a	a	a	b	a	a	b	b	b

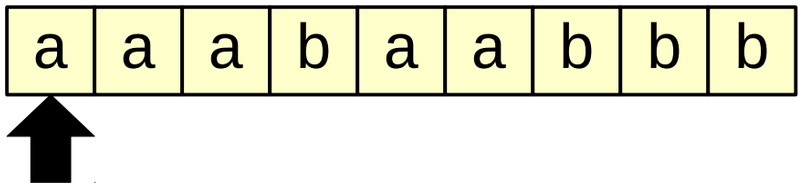
Generalizing NFAs





Generalizing NFAs





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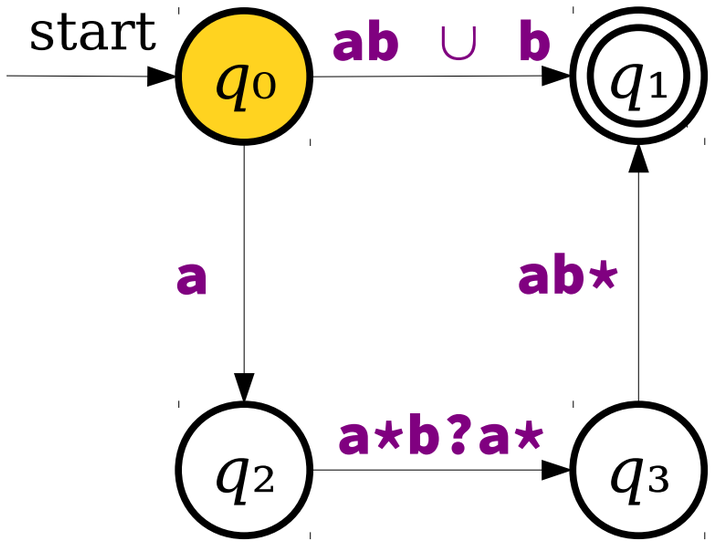

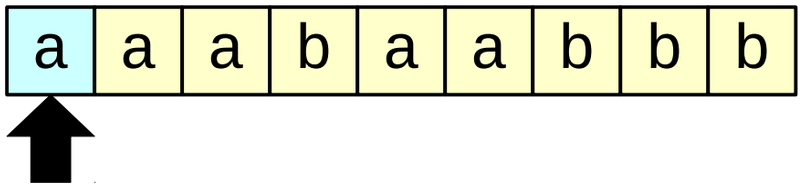

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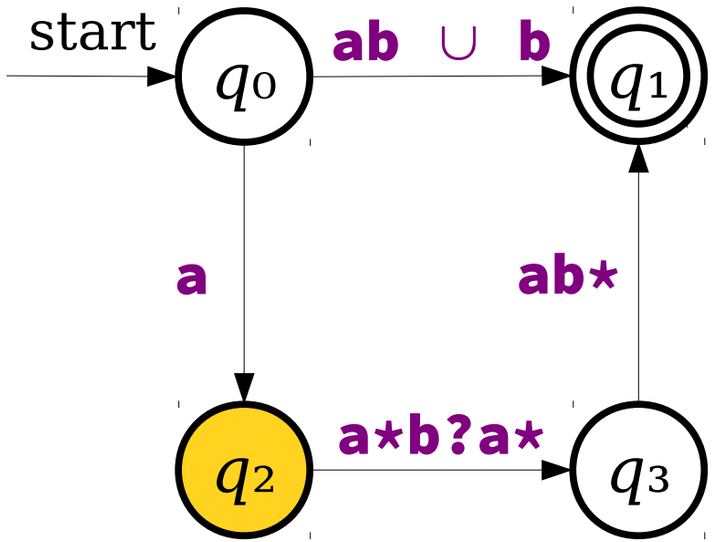

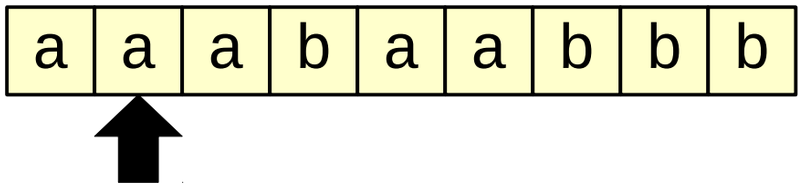

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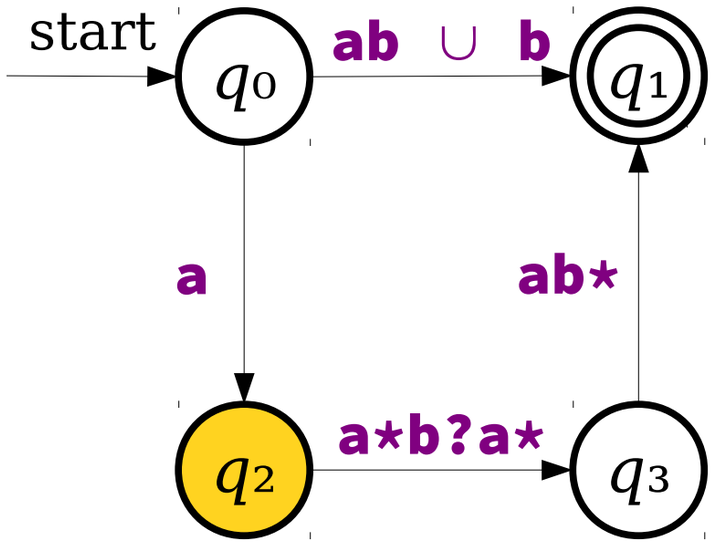
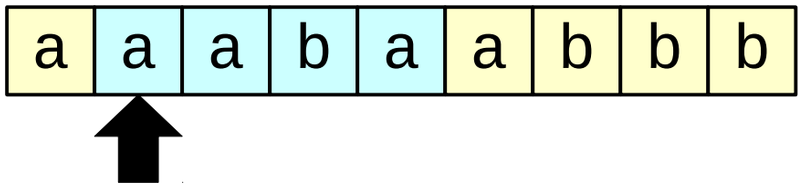
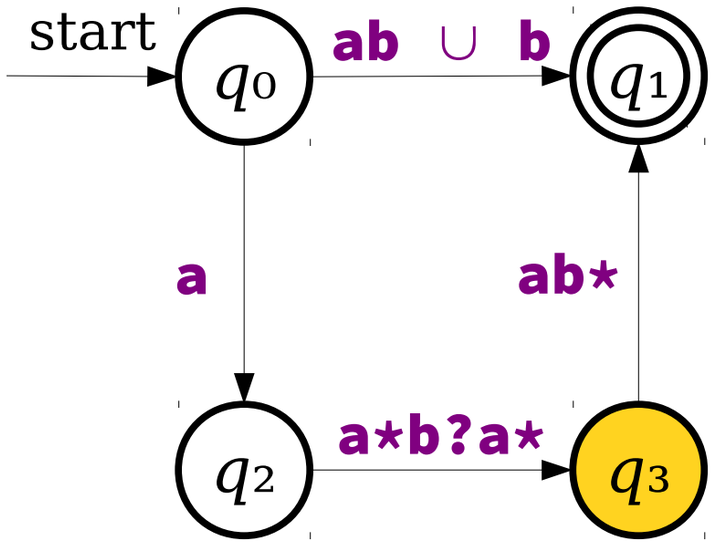
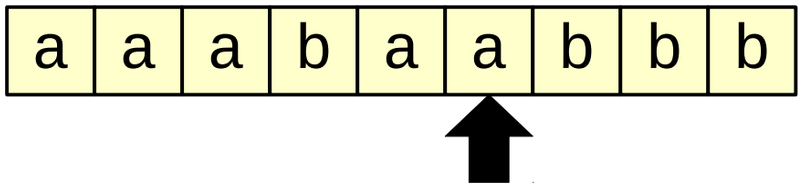


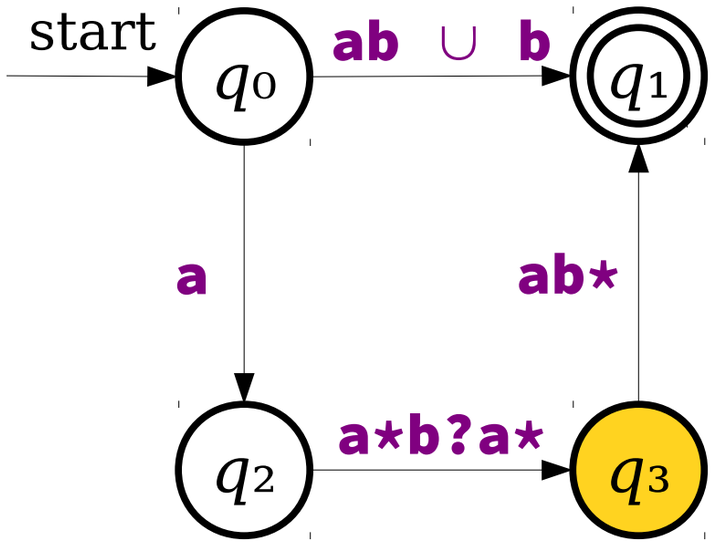


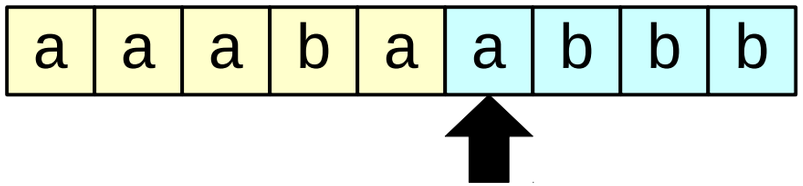
Generalizing NFAs





Generalizing NFAs





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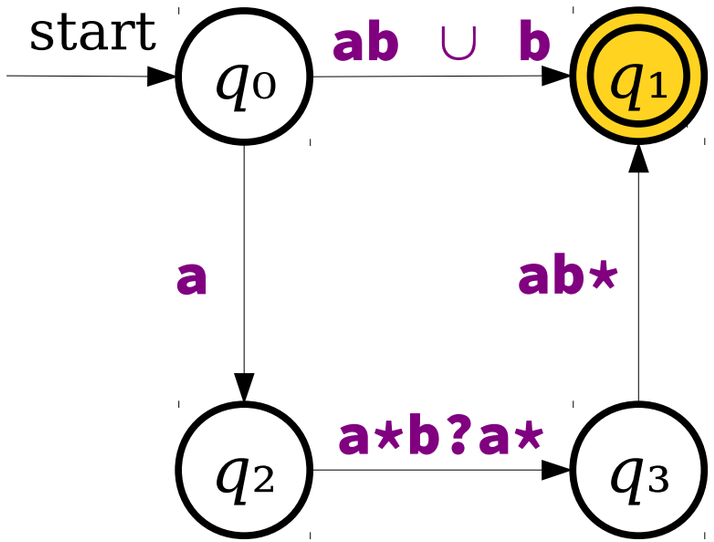
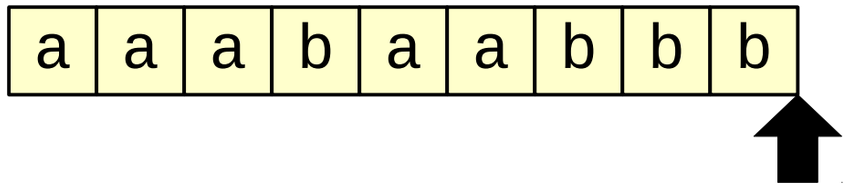




Key Idea 1: Imagine that we can label transitions in an NFA with arbitrary regular expressions.

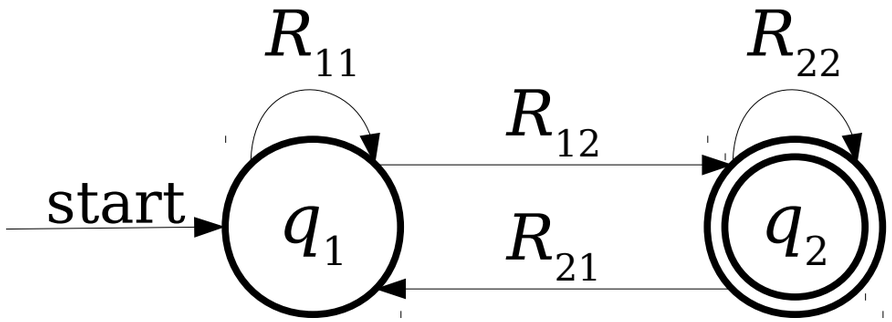
Generalizing NFAs

Key Idea 2: If we can convert an NFA into a generalized NFA that looks like this...

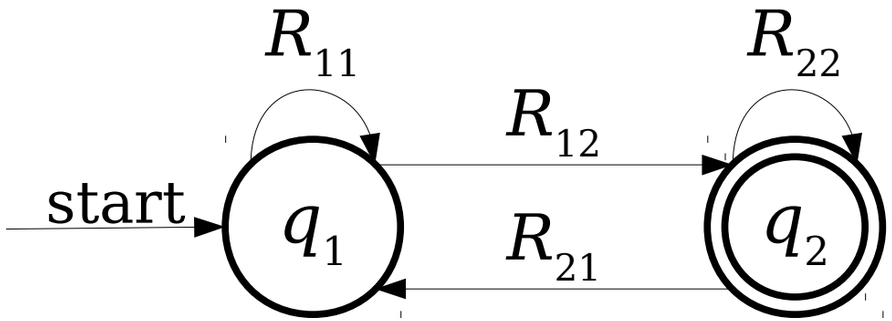
...then we can easily read off a regular expression for the original NFA.

...but not all NFAs look like this :-(
but maybe we can fix that?

From NFAs to Regular Expressions

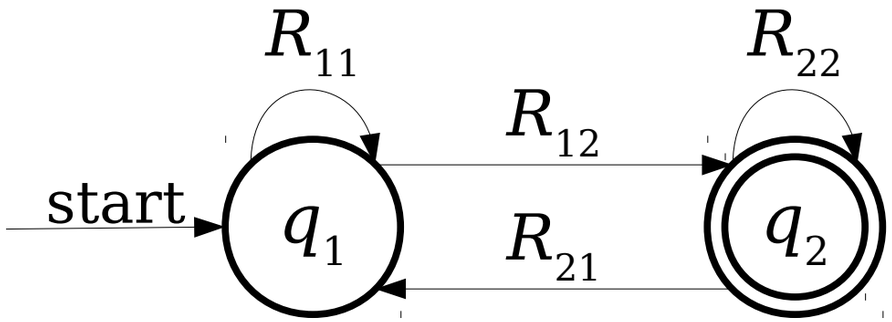


From NFAs to Regular Expressions



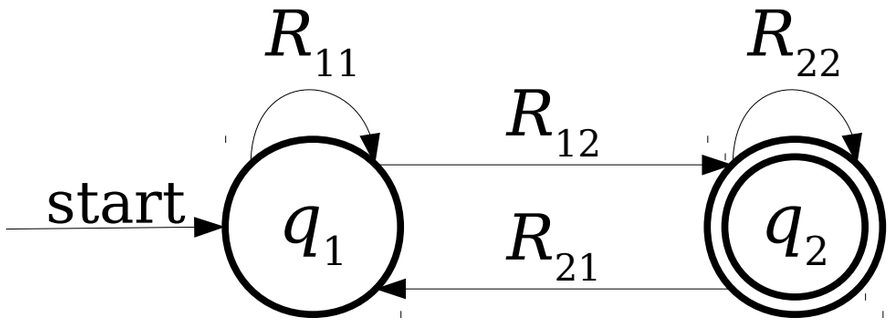
Here, R_{11}, R_{12}, R_{21}, and R_{22} are arbitrary regular expressions.

From NFAs to Regular Expressions



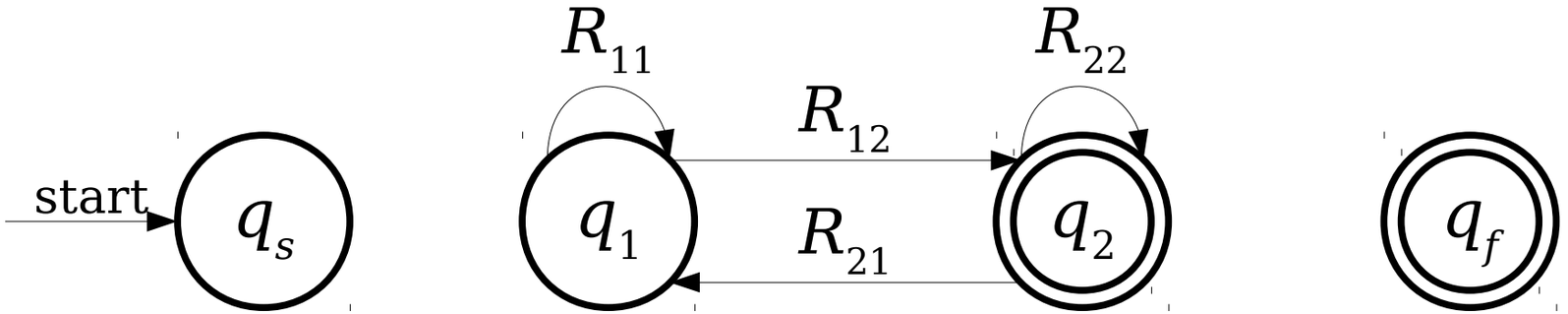
Key Idea 2 Goal: Somehow transform this NFA so that it looks like this:

From NFAs to Regular Expressions



The first step is going to be a simple one...

From NFAs to Regular Expressions



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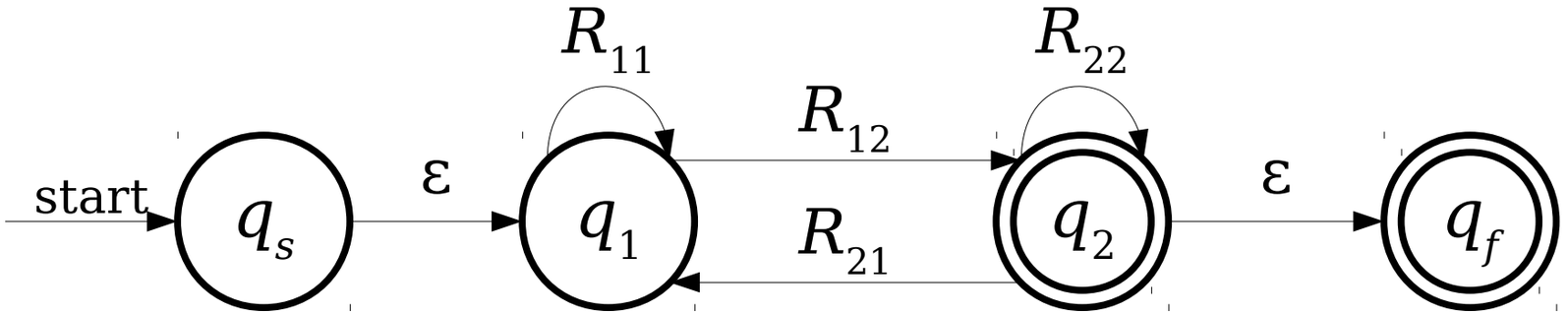

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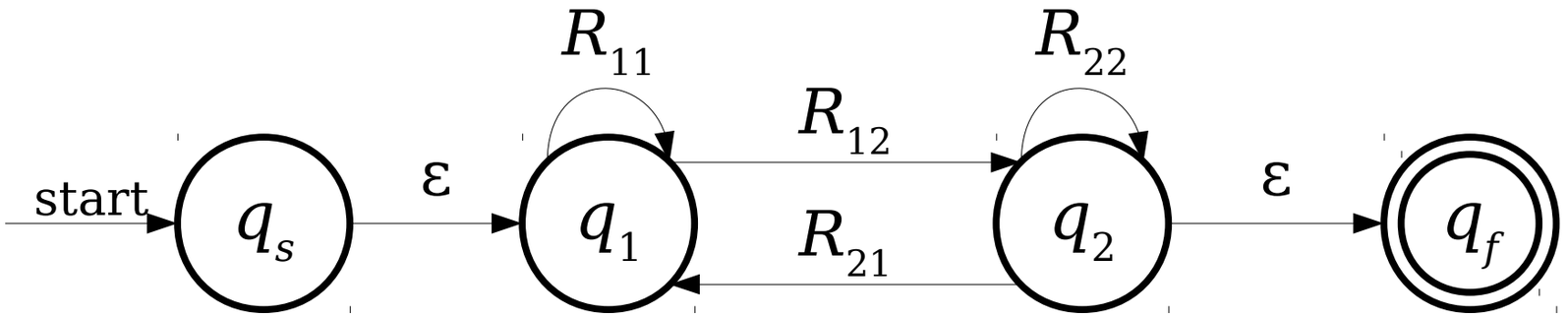
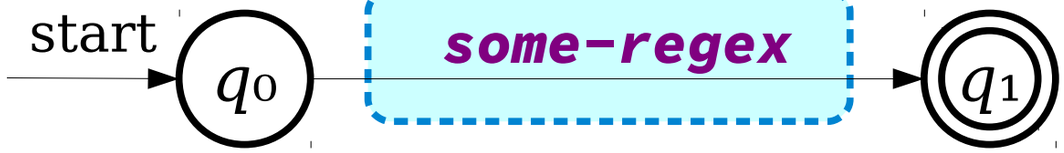


Key Idea 2 Goal Progress: Now it has clean start and end, but middle is messy. Need to get rid of those states.



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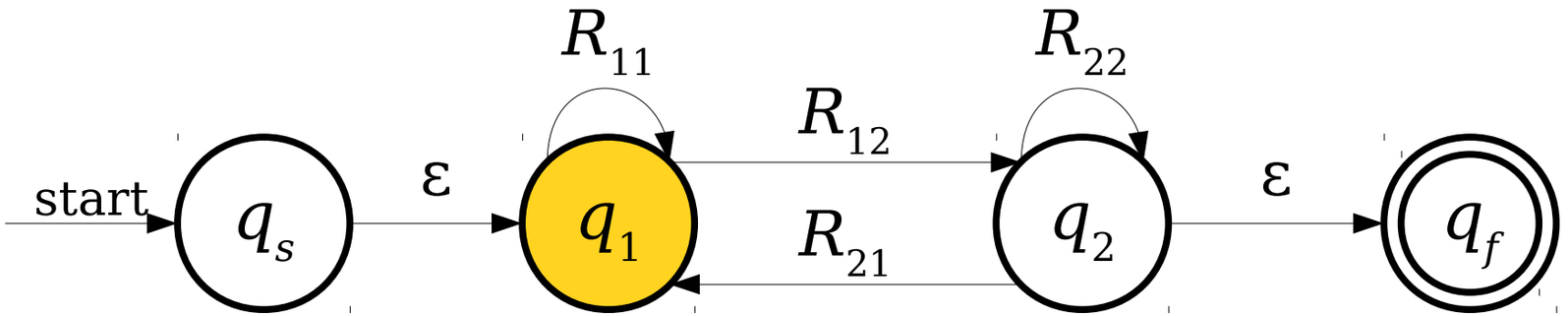

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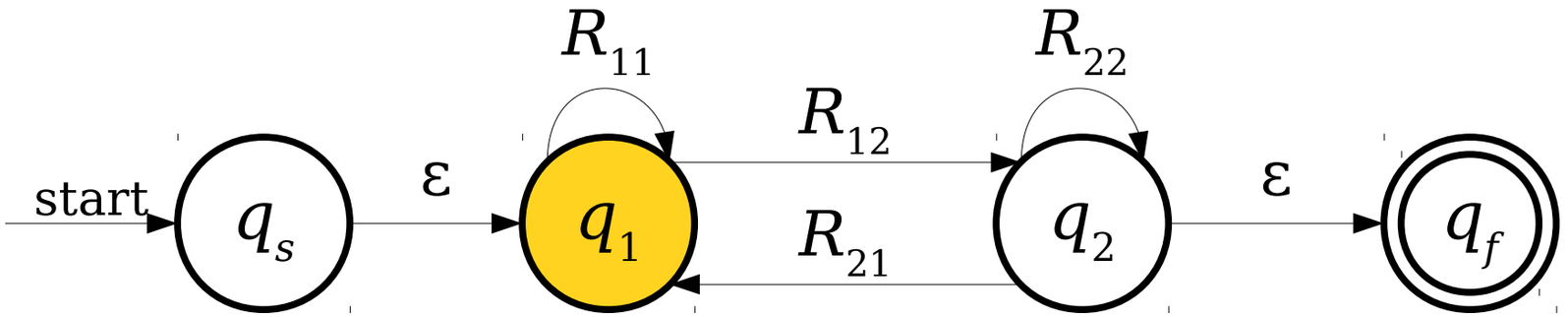
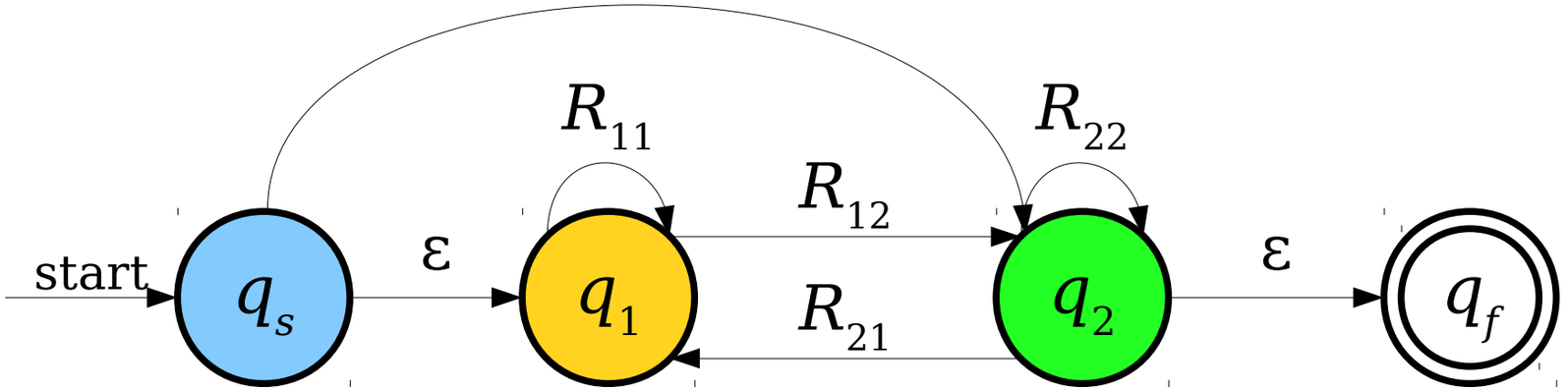


Could we eliminate this state from the NFA?

From NFAs to Regular Expressions



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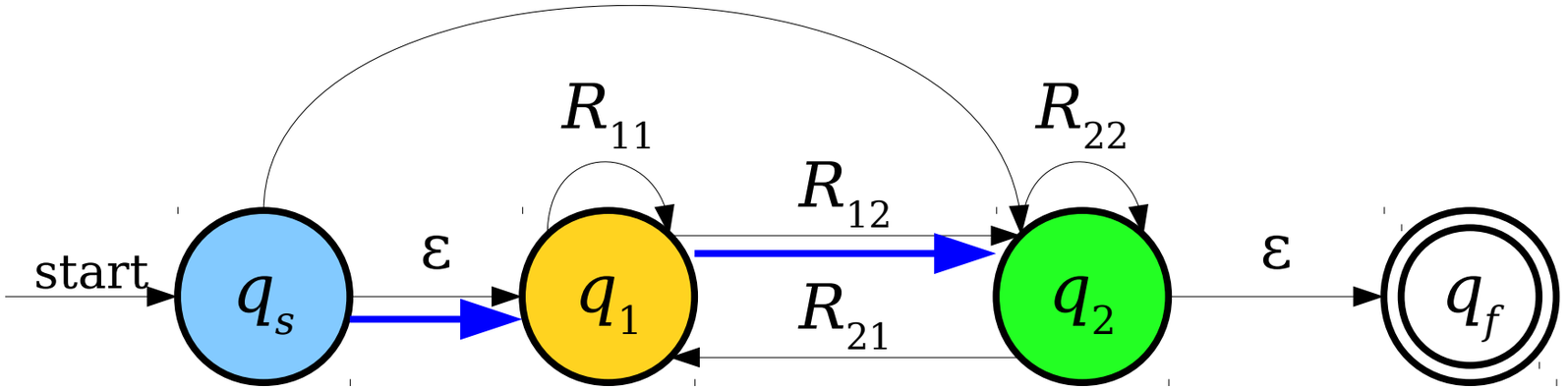
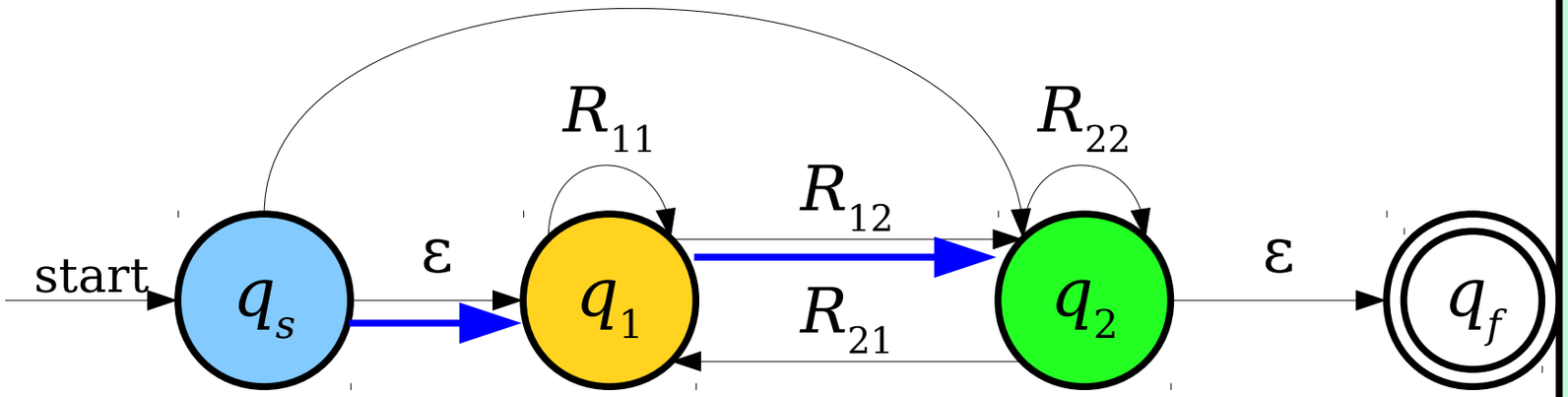


Here is a pattern that we might process when going from q_s to q_2: εR_{12}

From NFAs to Regular Expressions



Here is a pattern that we might process when going from q_s to q_2: εR_{12}

State elimination question:

How many of the following are also patterns might we process when going from q_s to q_2?

$\varepsilon R_{11} R_{12}$

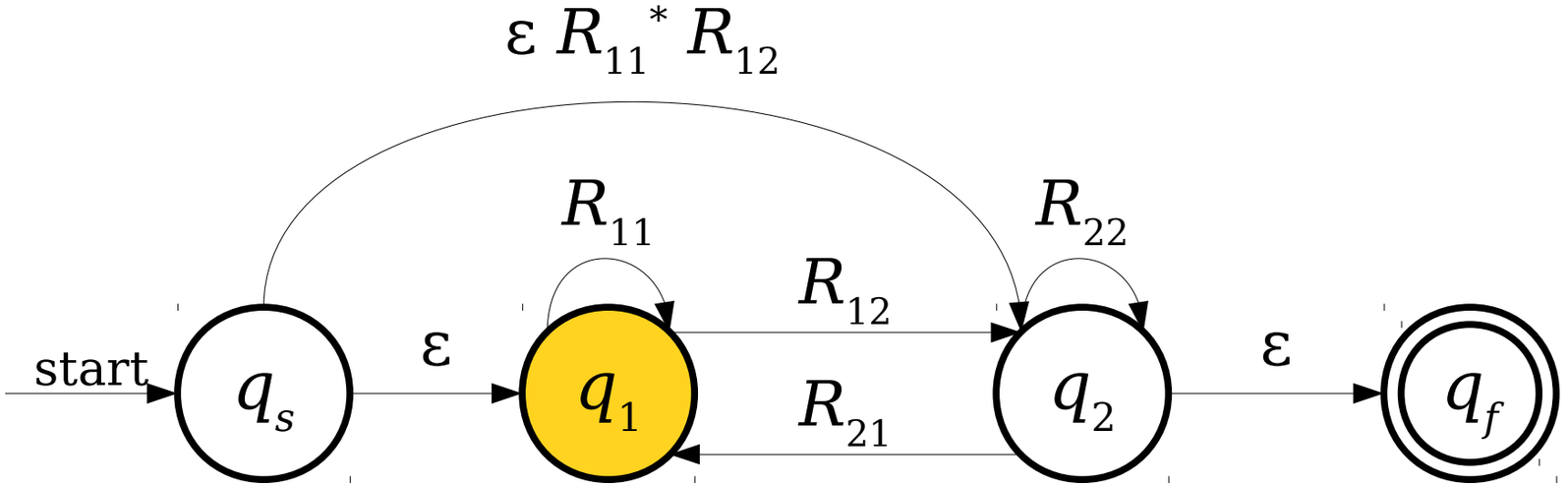
$\varepsilon R_{11} R_{11} R_{12}$

$\varepsilon R_{11} R_{12} R_{11}$

$\varepsilon R_{11} R_{11} R_{12}$

$\varepsilon R_{11} R_{11} R_{11} R_{11} R_{11} R_{12}$

From NFAs to Regular Expressions



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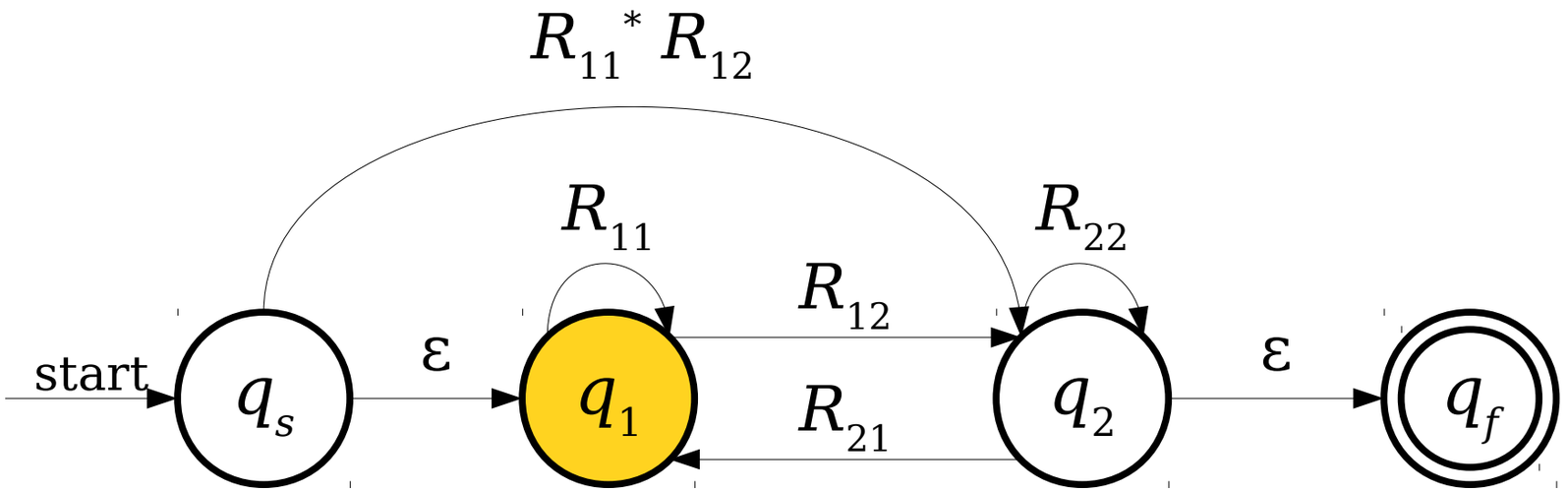

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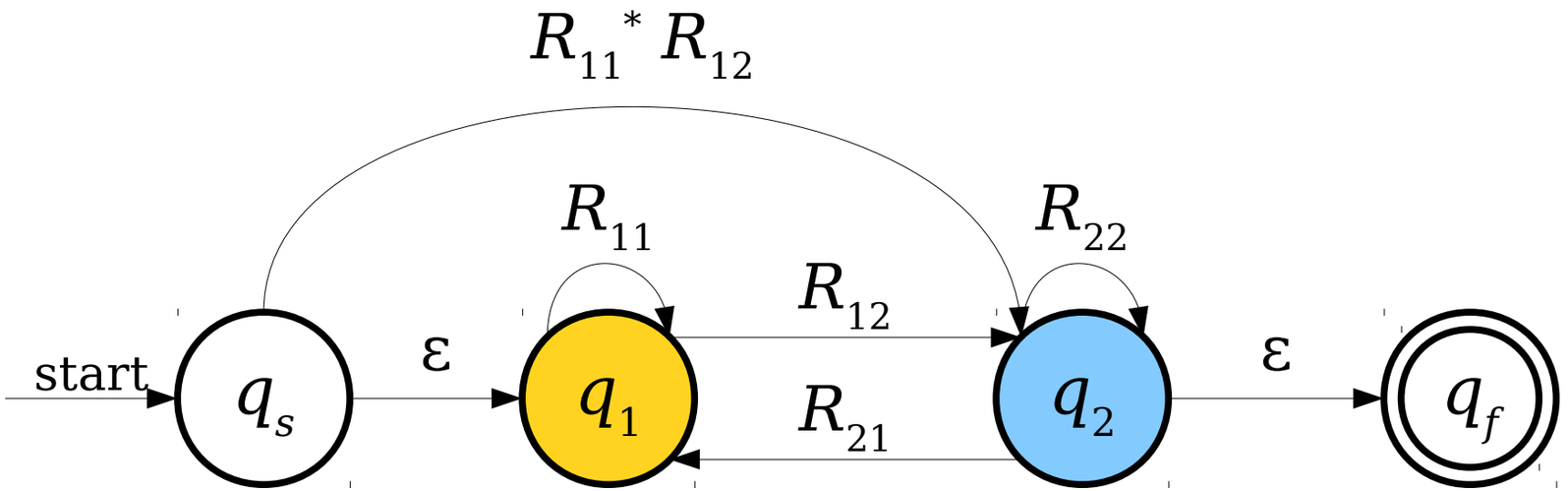

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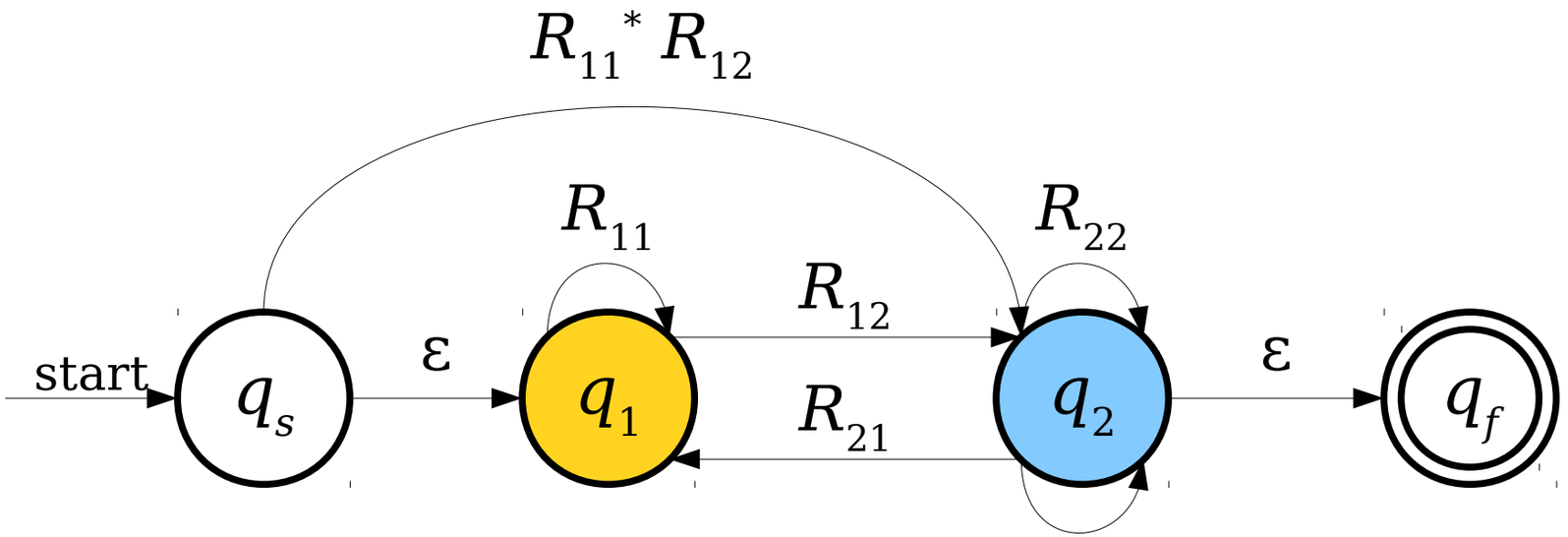

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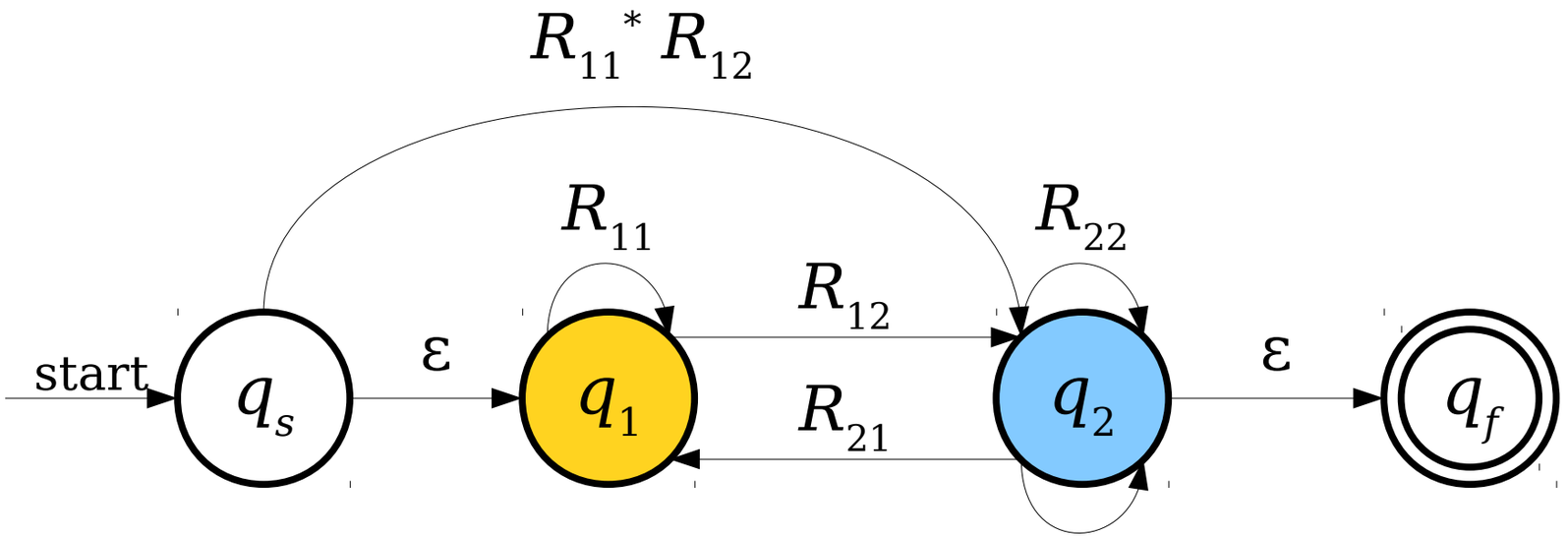
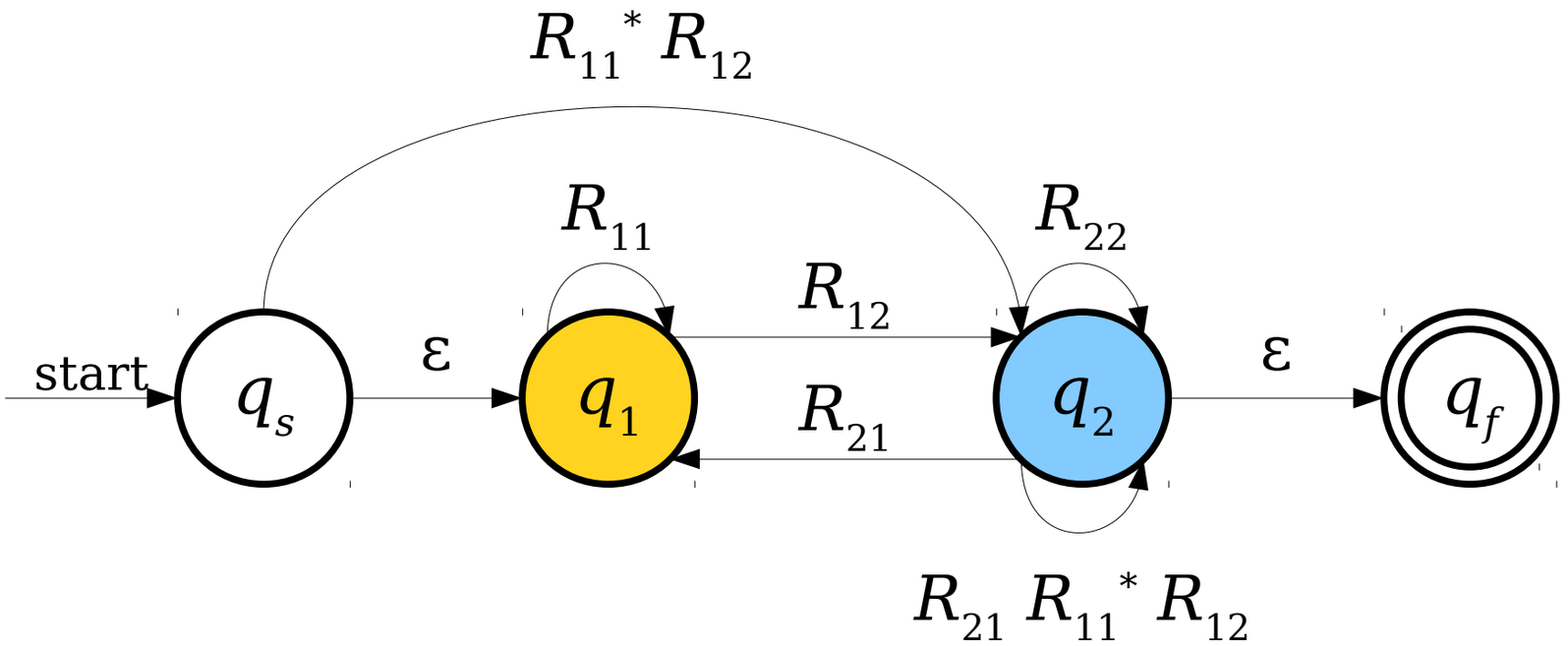


Here is a pattern that we might process when going from q_2 to q_2: $R_{21}R_{11}R_{12}$

From NFAs to Regular Expressions



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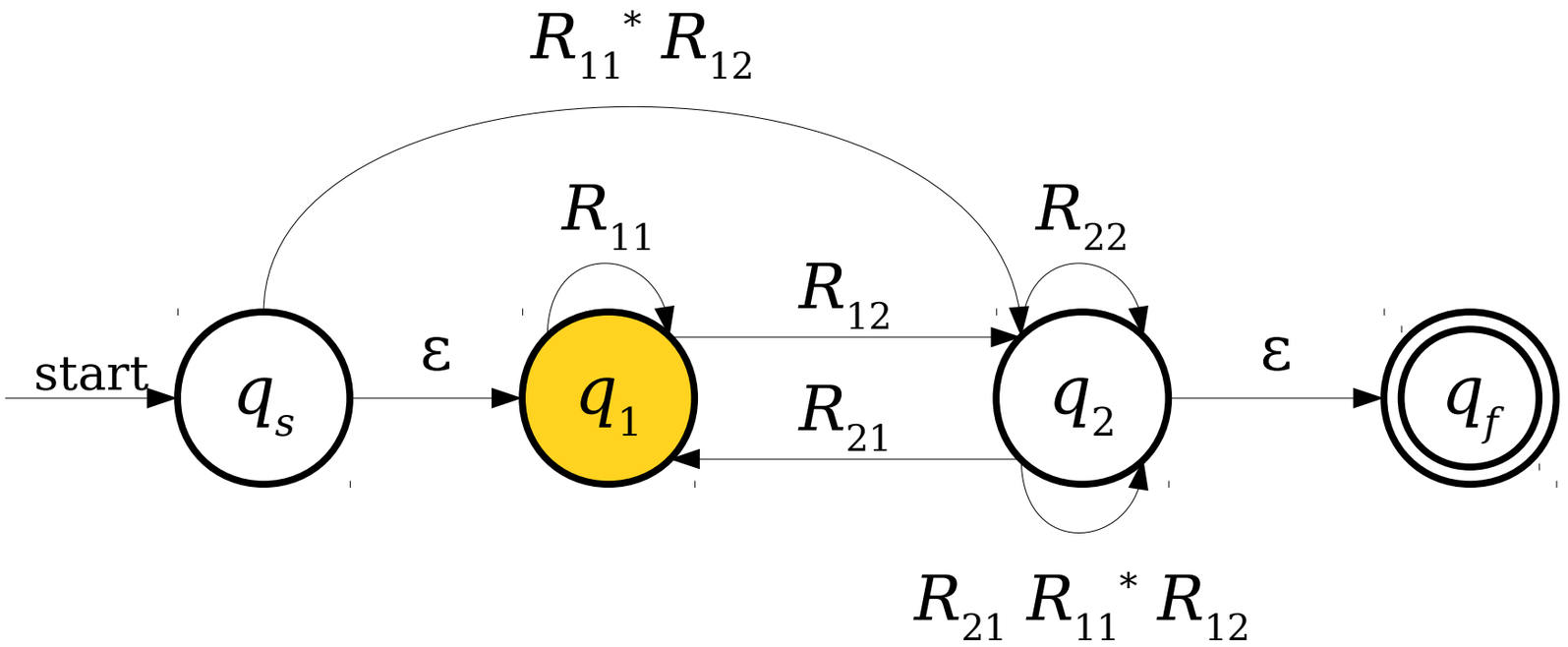

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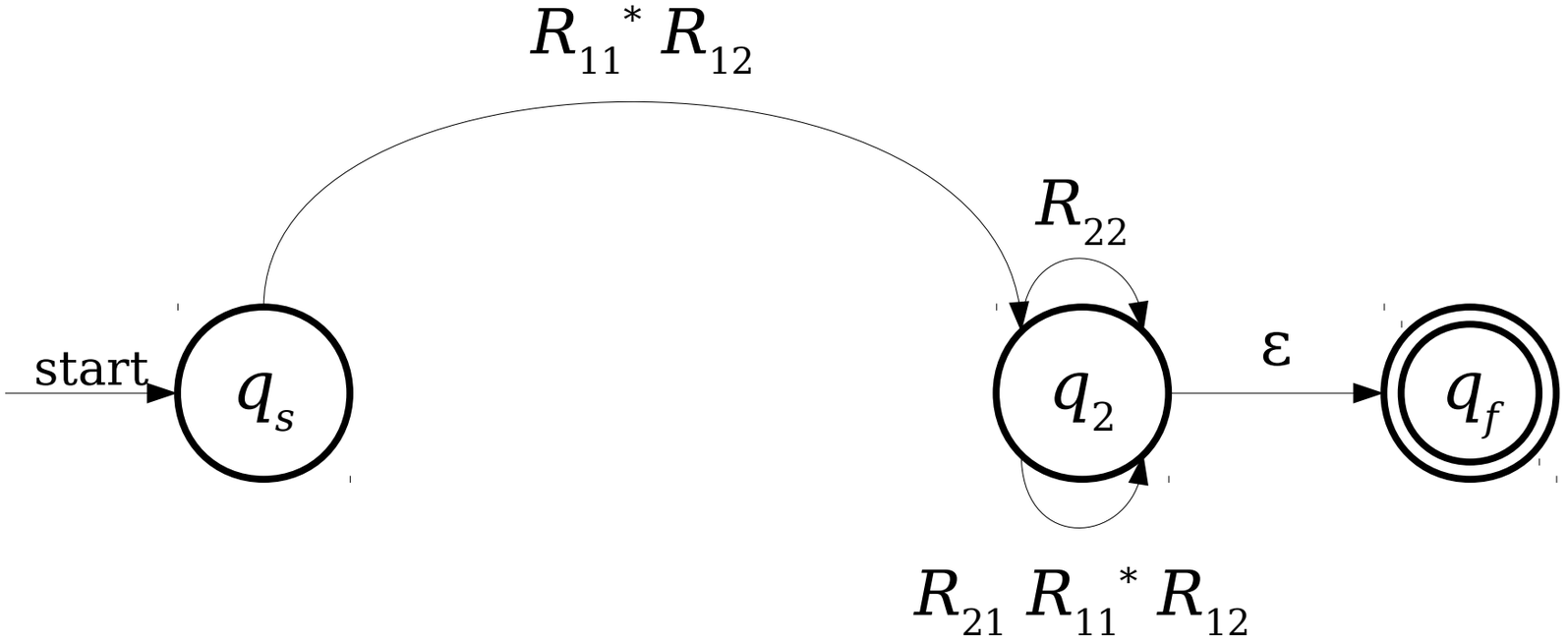

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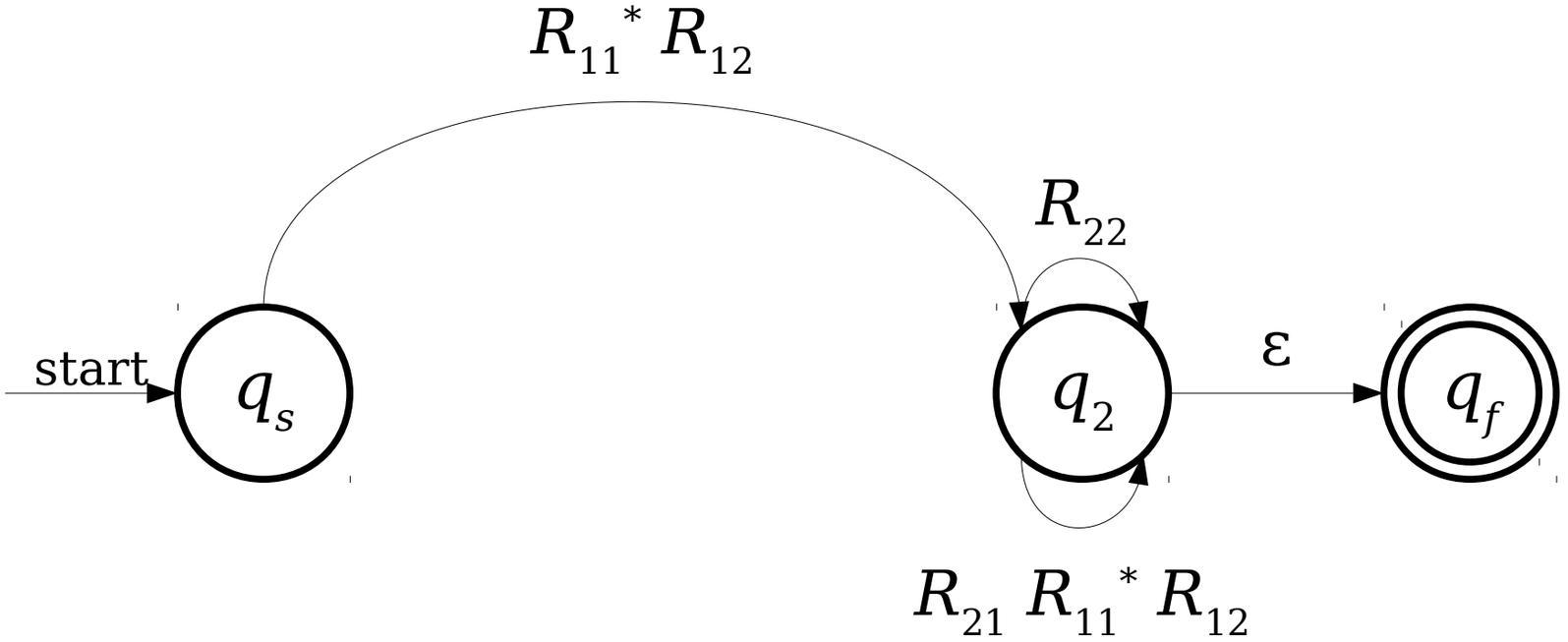

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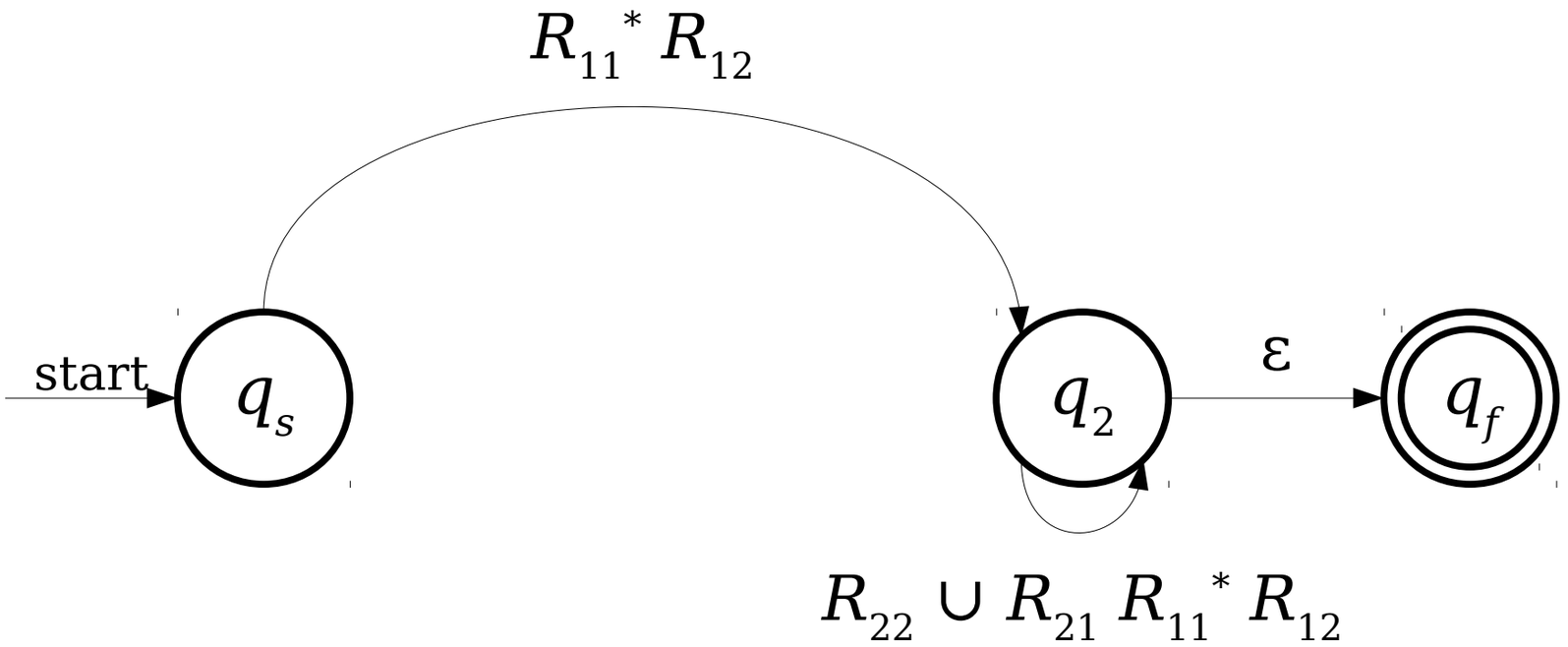
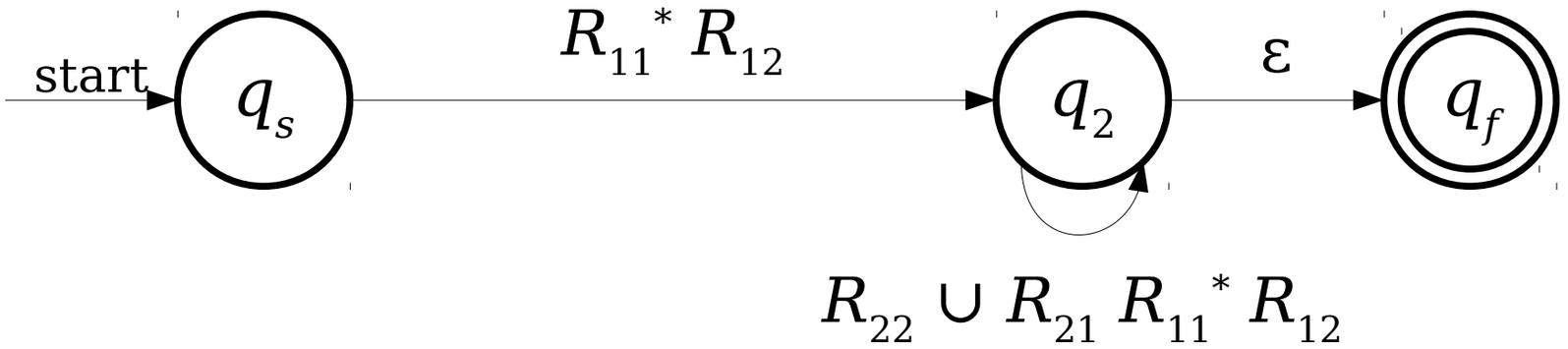


Note: We're using **union** to combine these transitions together.

From NFAs to Regular Expressions



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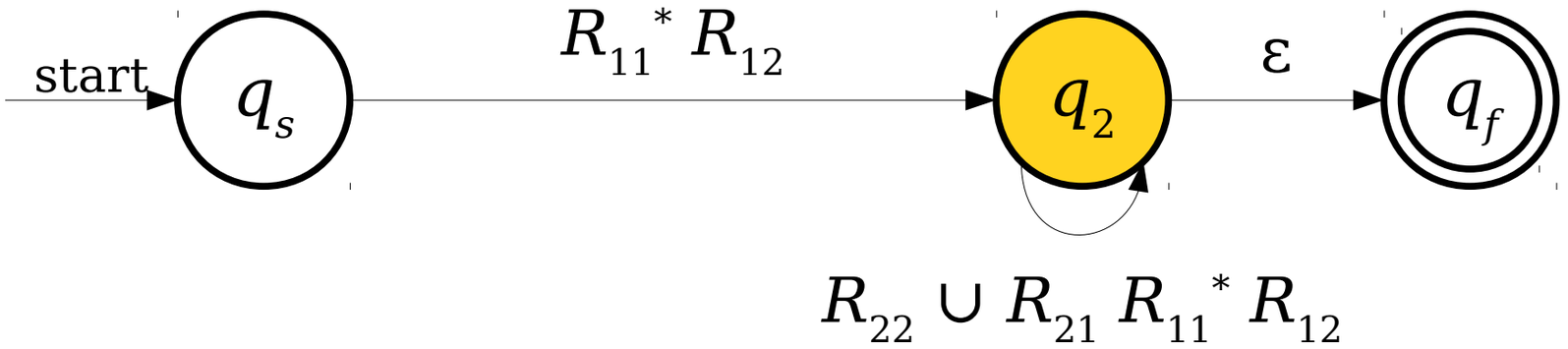

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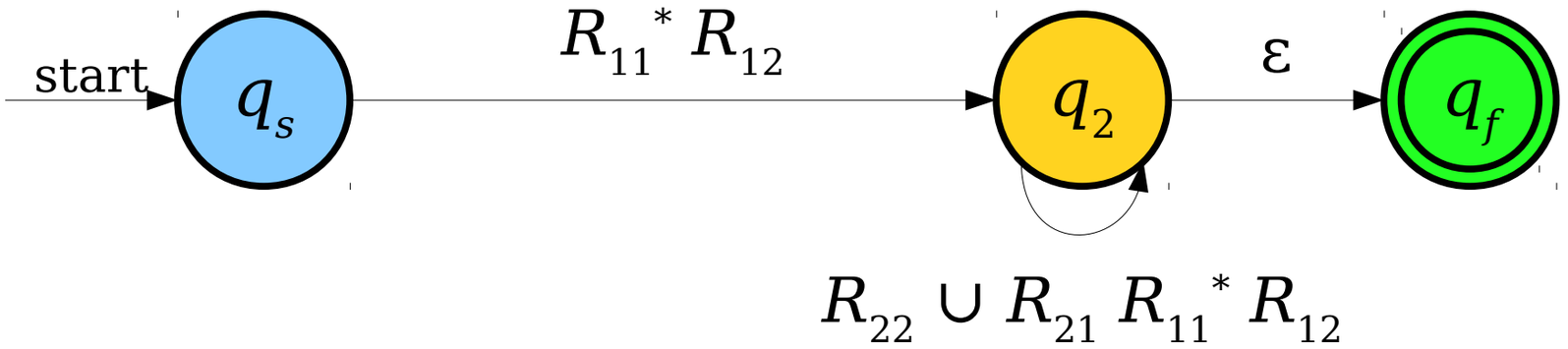

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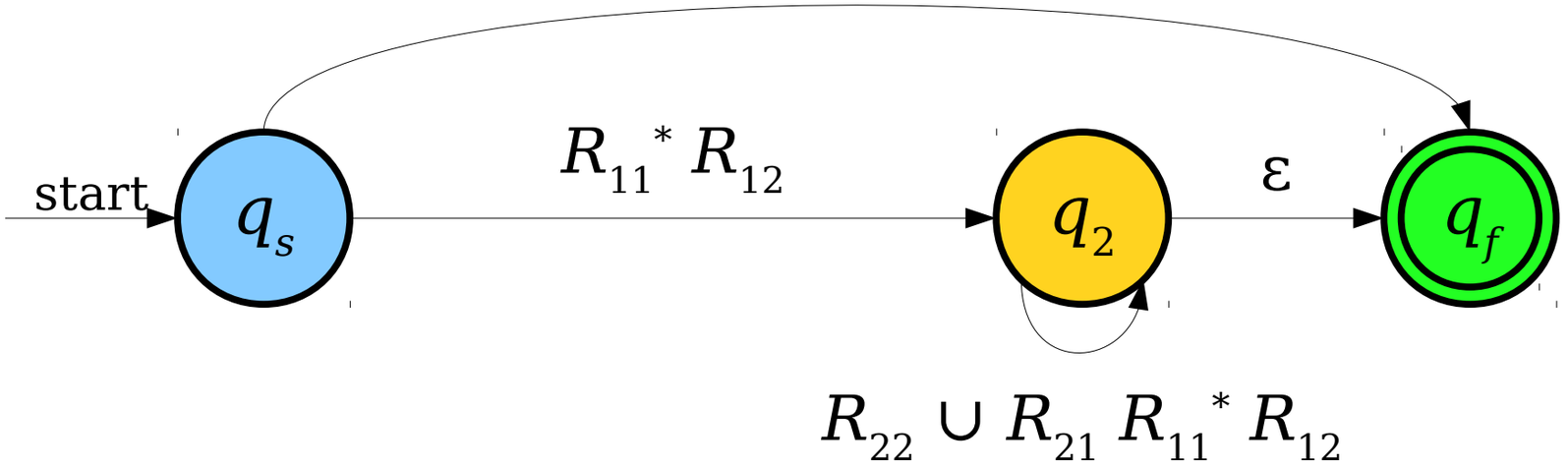
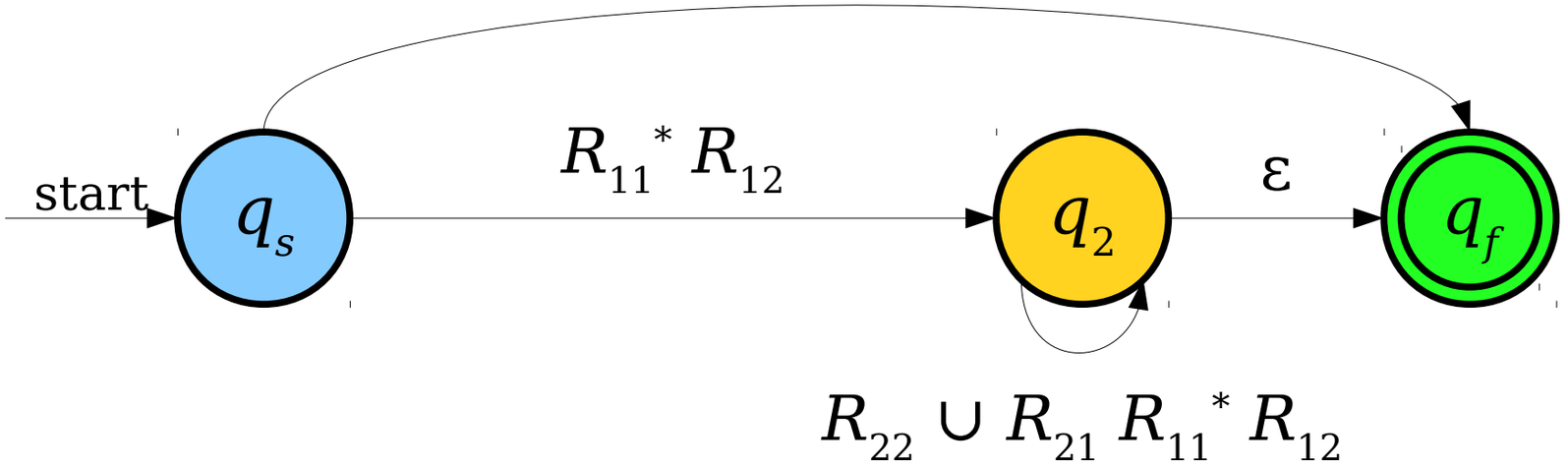


From NFAs to Regular Expressions

Quick check: what goes on this transition?



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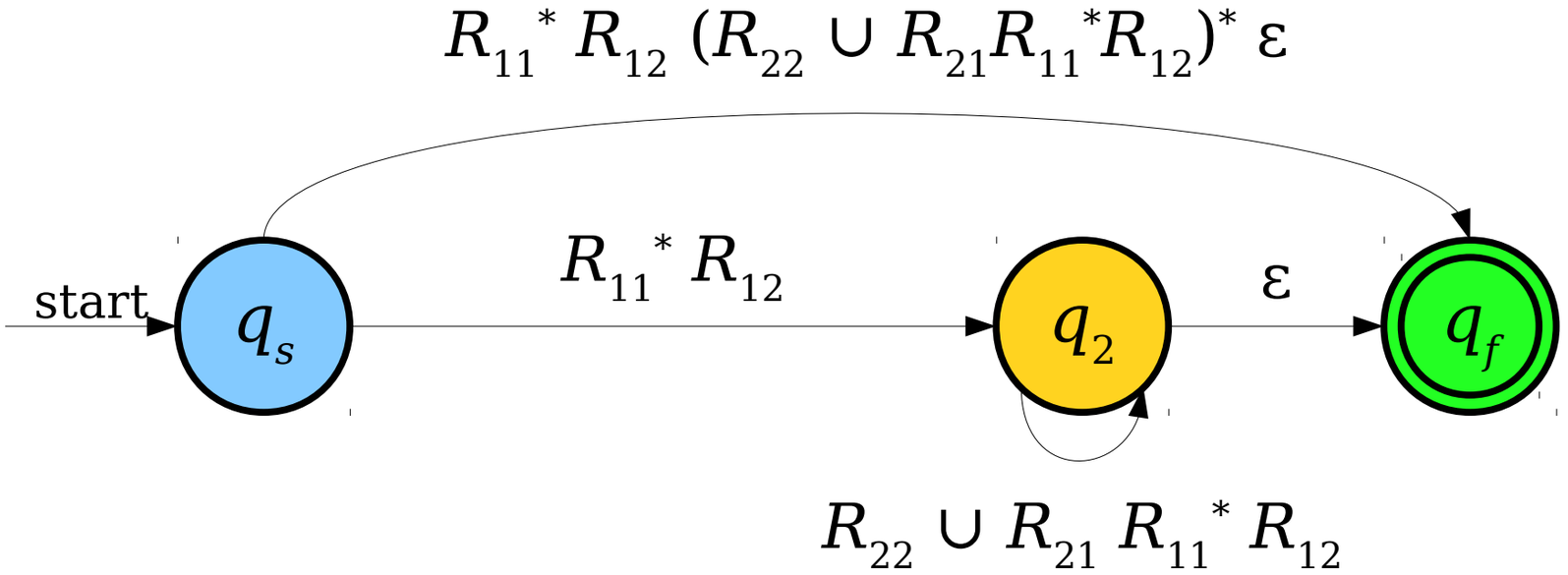

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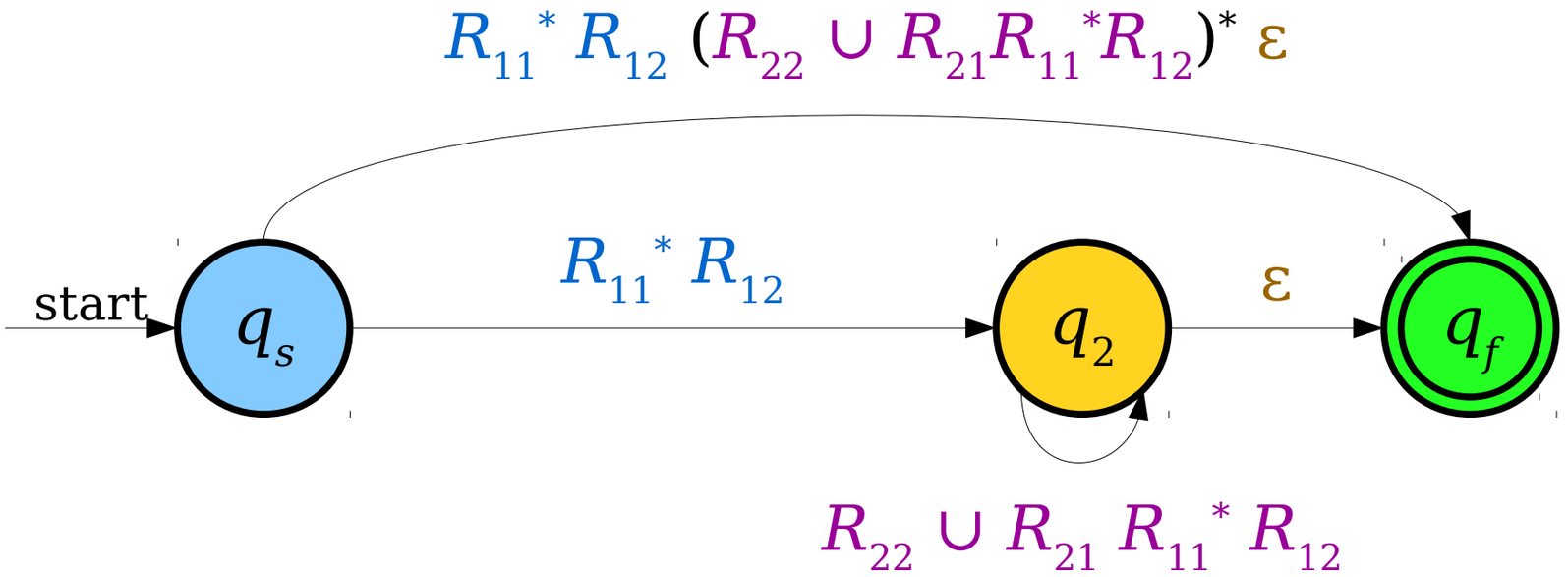
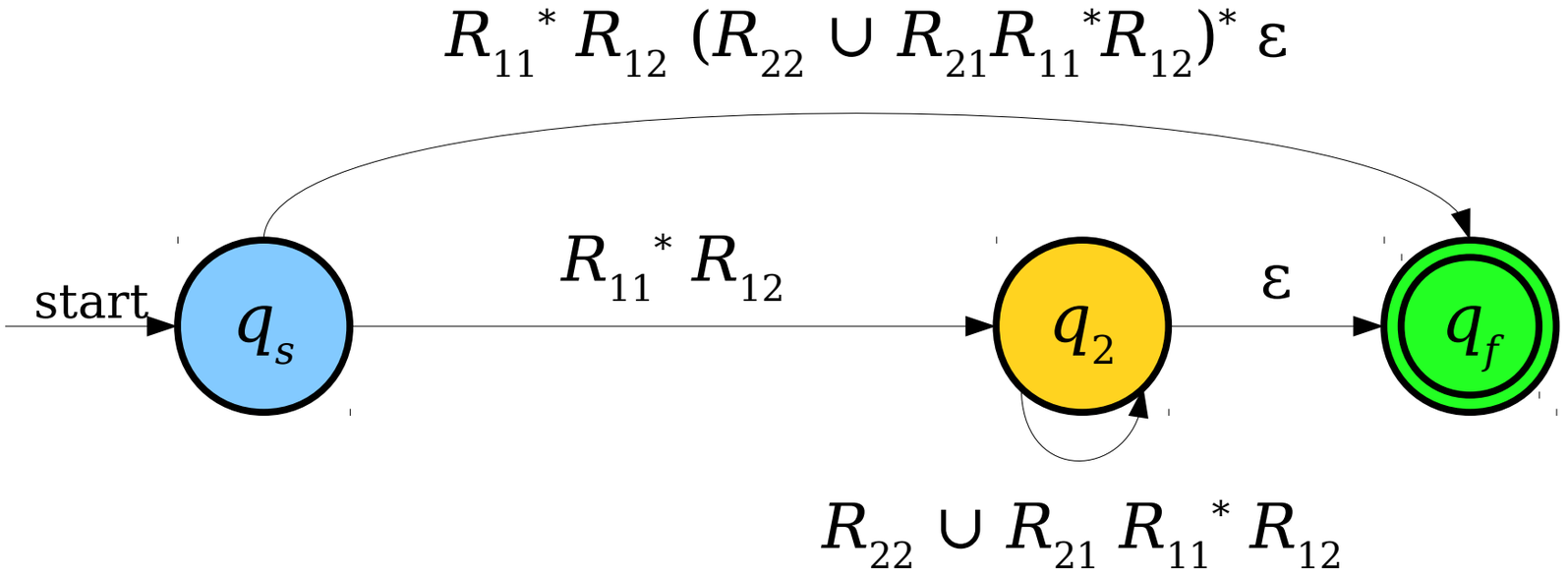
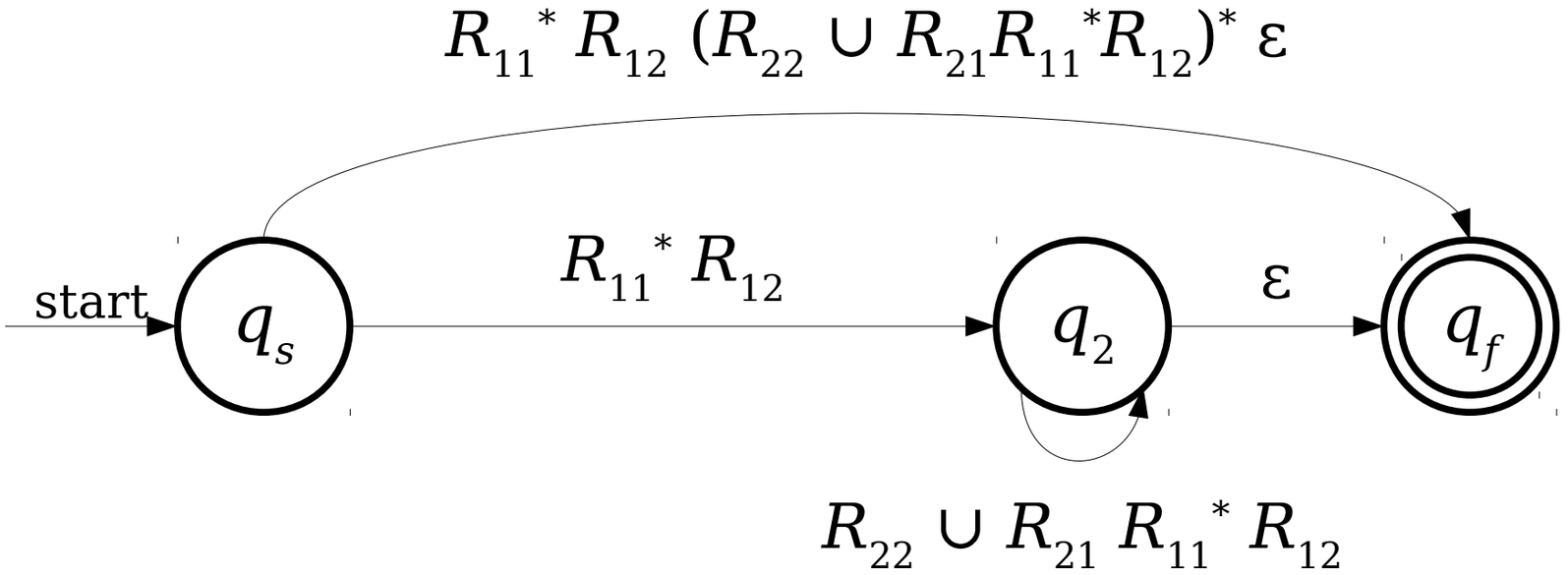


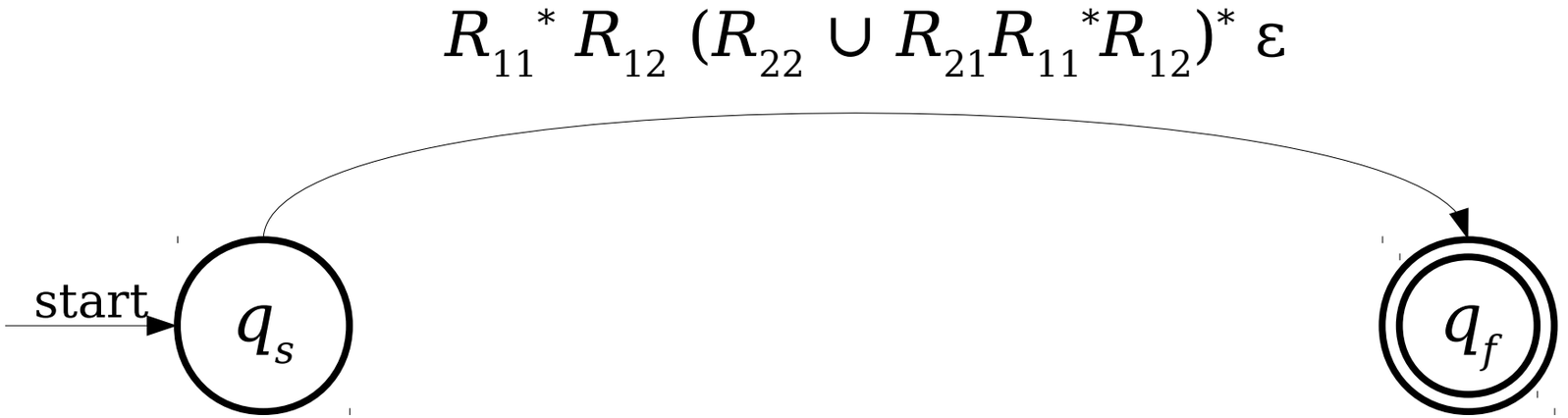
From NFAs to Regular Expressions



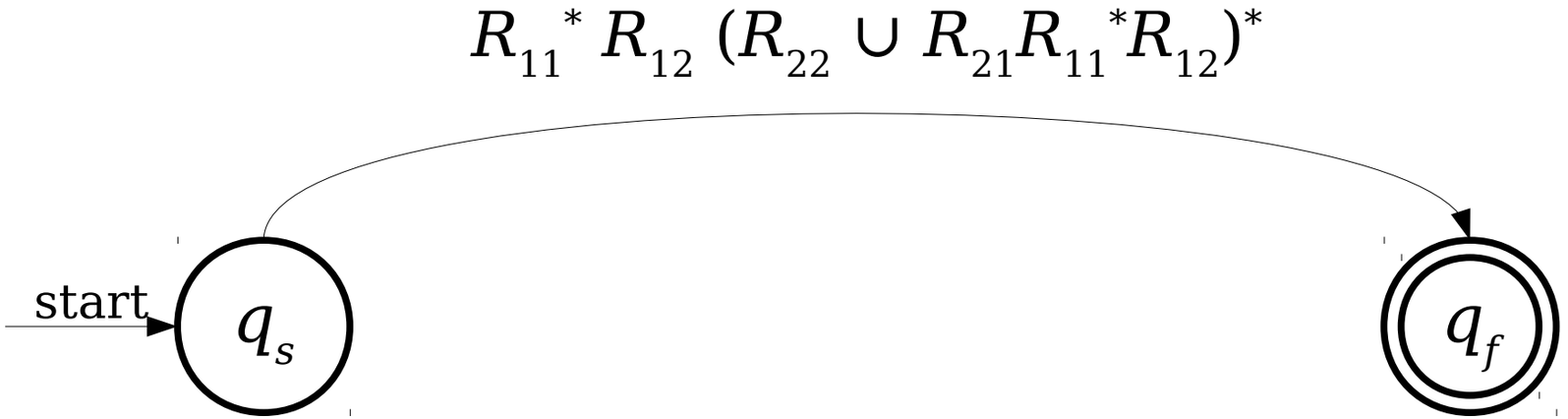
From NFAs to Regular Expressions



From NFAs to Regular Expressions



From NFAs to Regular Expressions



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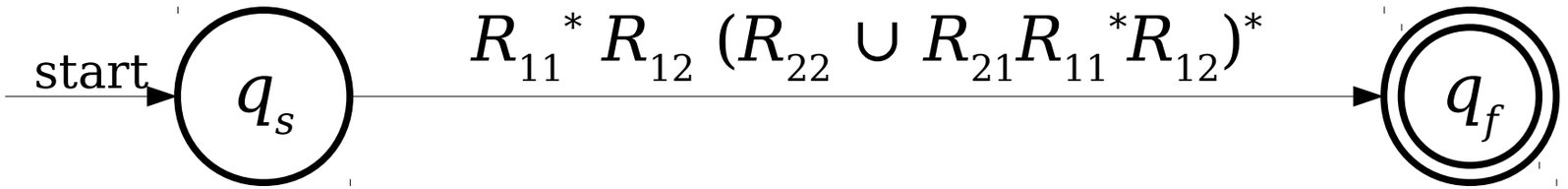
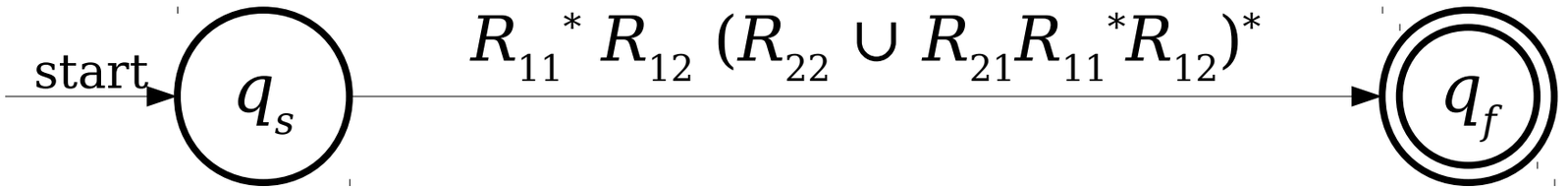
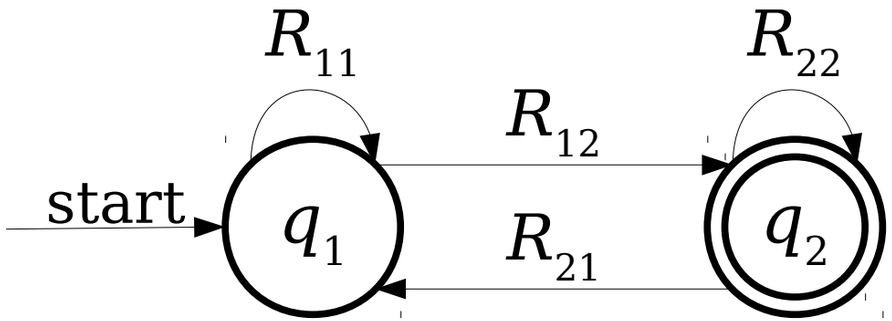


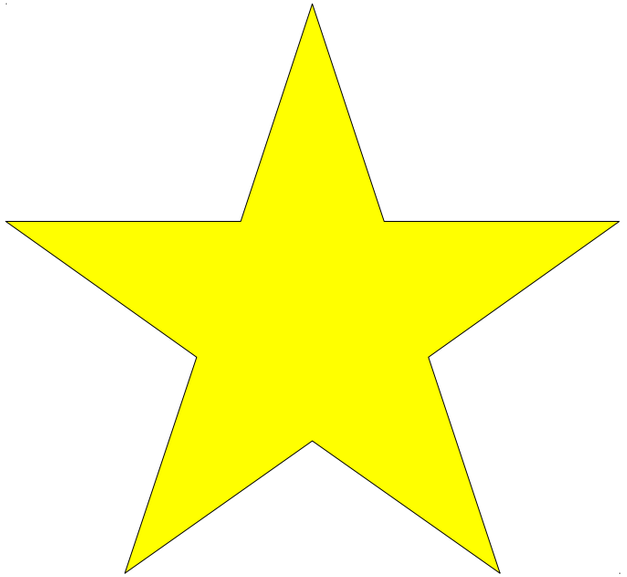
From NFAs to Regular Expressions



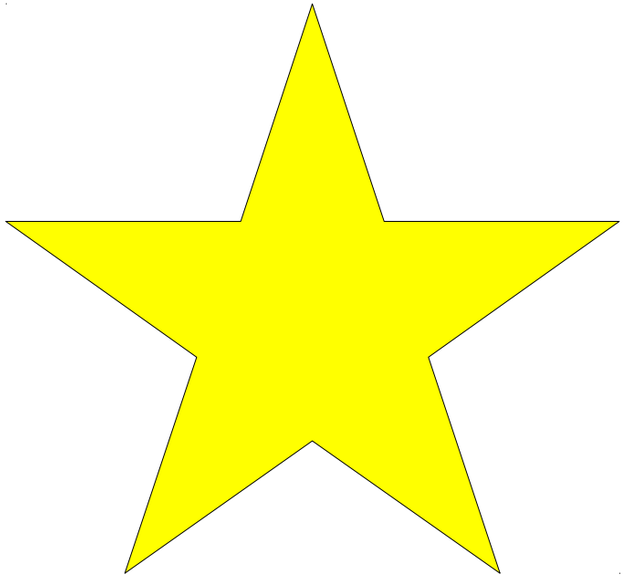


The State-Elimination Algorithm

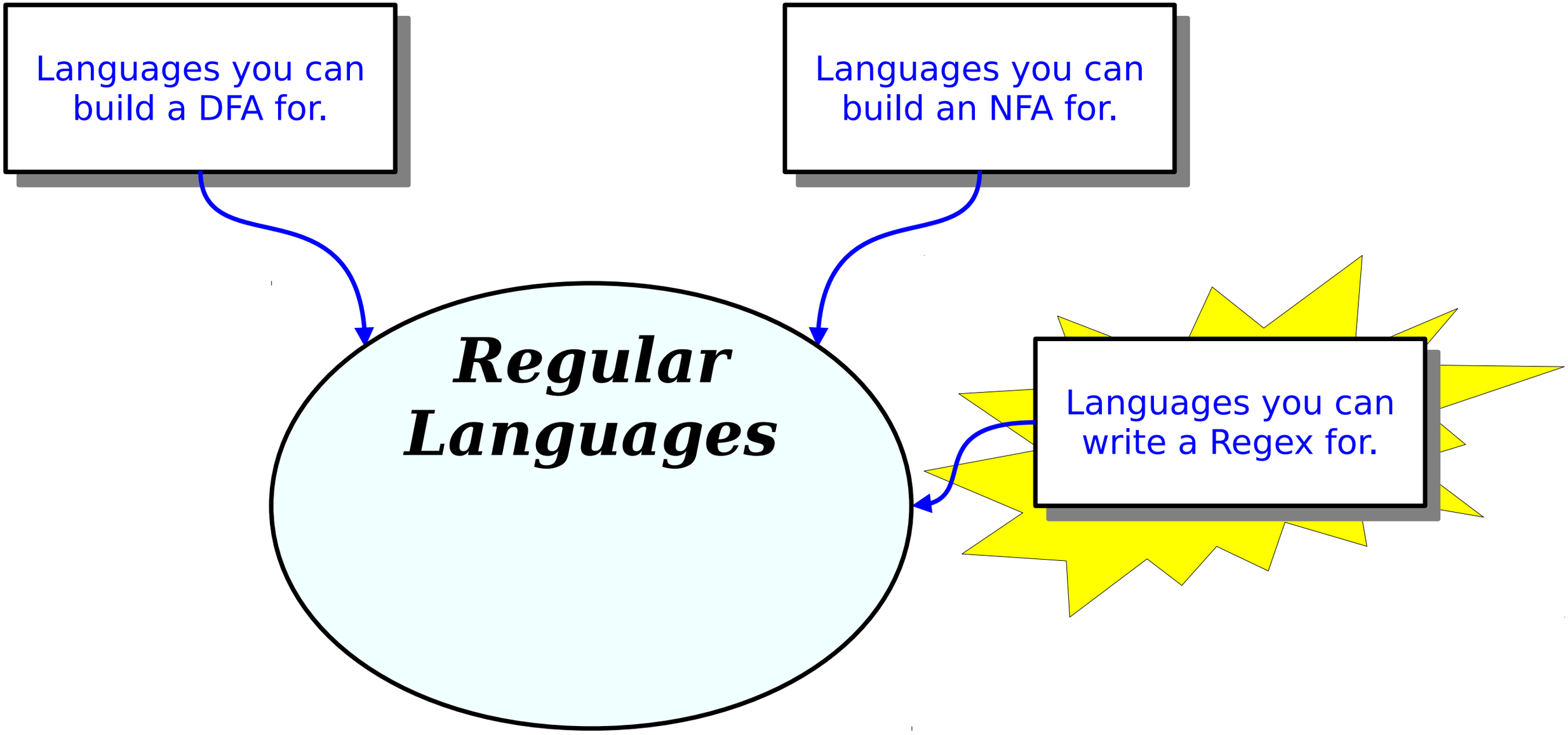
- Start with an NFA N for the language L.
- Add a new start state q_s and accept state q_f to the NFA.
  - Add an ε-transition from q_s to the old start state of N.
  - Add ε-transitions from each accepting state of N to q_f, then mark them as not accepting.
- Repeatedly remove states other than q_s and q_f from the NFA by "shortcutting" them until only two states remain: q_s and q_f.
- The transition from q_s to q_f is then a regular expression for the NFA.



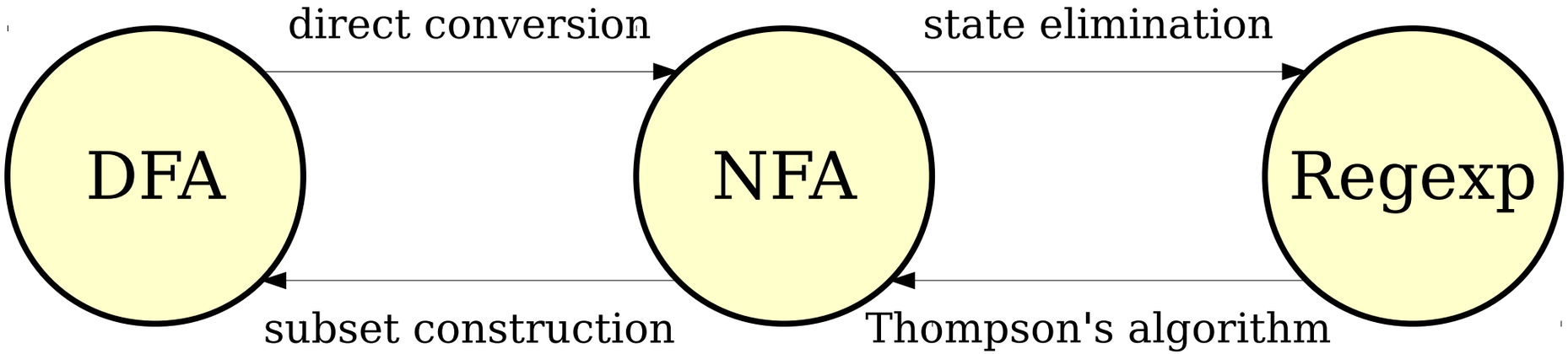
The State-Elimination Algorithm



- To eliminate a state q from the automaton, do the following for each pair of states q_0 and q_1, where there's a transition from q_0 into q and a transition from q into q_1:
  - Let R_{in} be the regex on the transition from q_0 to q.
  - Let R_{out} be the regex on the transition from q to q_1.
  - If there is a regular expression R_{stay} on a transition from q to itself, add a new transition from q_0 to q_1 labeled $((R_{in})(R_{stay})^*(R_{out}))$.
  - If there isn't, add a new transition from q_0 to q_1 labeled $((R_{in})(R_{out}))$
- If a pair of states has multiple transitions between them labeled R_1, R_2, …, R_k, replace them with a single transition labeled $R_1 \cup R_2 \cup \ldots \cup R_k$.


Languages you can build a DFA for.
Languages you can build an NFA for.
Regular Languages
Languages you can write a Regex for.

Our Transformations



Theorem: The following are all equivalent:

- L is a regular language.
- There is a DFA D such that $\mathscr{L}(D) = L$.
- There is an NFA N such that $\mathscr{L}(N) = L$.
- There is a regular expression R such that $\mathscr{L}(R) = L$.

Next Time

- ***On Beyond Regular!***
  - What kinds of languages *aren't* regular?
- ***The Myhill-Nerode Theorem***
  - Proof technique to show non-regular